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Fivebrane Instanton Corrections to the Universal Hypermultiplet

Marijn Davidse[1], Ulrich Theis[2] and Stefan Vandoren[1]

[1]*Institute for Theoretical Physics* and *Spinoza Institute*
Utrecht University, 3508 TD Utrecht, The Netherlands
M.Davidse, S.Vandoren@phys.uu.nl

[2]*Institute for Theoretical Physics, Vienna University of Technology,*
Wiedner Hauptstrasse 8–10, A-1040 Vienna, Austria
theis@hep.itp.tuwien.ac.at

Abstract

We analyze the Neveu-Schwarz fivebrane instanton in type IIA string theory compactifications on rigid Calabi-Yau threefolds, in the low-energy supergravity approximation. It there appears as a finite action solution to the Euclidean equations of motion of a double-tensor multiplet (dual to the universal hypermultiplet) coupled to $N = 2$, $D = 4$ supergravity. We determine the bosonic and fermionic zero modes, and the single-centered instanton measure on the moduli space of collective coordinates. The results are then used to compute, in the semiclassical approximation, correlation functions that nonperturbatively correct the universal hypermultiplet moduli space geometry of the low-energy effective action. We find that only the Ramond-Ramond sector receives corrections, and we discuss the breaking of isometries due to instantons.

Contents

1 Introduction

One particular object that plays an important role in nonperturbative string theory is the Neveu-Schwarz fivebrane (NS5-brane). It is the magnetic dual of the fundamental string, and appears as a soliton-like solution preserving half of the supersymmetry, for instance in heterotic [1] and type II strings [2]. Compared to the type IIB theory, the NS5-brane in IIA is particularly intriguing because, unlike D-branes, strings cannot end on it. This makes it difficult to construct the worldvolume theory, or to compute the partition function of the fivebrane, see e.g. [3] and references therein.

The transverse space to the fivebrane is four-dimensional with coordinates x^μ, $\mu = 1, \ldots, 4$, and as a solution to the supergravity equations of motion, it is characterized by the dilaton and the NS 2-form field strength,

$$\mathrm{e}^{2\phi} = \mathrm{e}^{2\phi_\infty} + \frac{Q}{r^2} \, , \qquad H_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho}{}^\sigma \, \nabla_\sigma \phi \, . \tag{1.1}$$

Here $r^2 = |x|^2$, and Q is the fivebrane charge, which is quantized in appropriate units. The four-dimensional spacetime metric is (conformally) flat, and we have set the gauge fields, present in the heterotic string, or any other fields from the Ramond-Ramond sector in type II, to zero. See [1, 2] for more details.

Clearly, the solution (1.1) is localized in the transverse four-dimensional Euclidean space, with the center chosen to be at the origin. It is easy to generalize (1.1) to multicentered solutions that are localized at a finite number of points in $\mathbb{R}^4$. From a four-dimensional point of view, one expects therefore the NS5-brane to be a (multi-centered) instanton. The remaining six dimensions, in the direction of the fivebrane, can be chosen non-compact and flat, but may also be replaced by a compact T^6, $K3 \times T^2$, or a Calabi-Yau (CY) threefold, around which the (Euclidean) fivebrane is wrapped. For theories with more than $N = 2$ supersymmetry in four dimensions, such instanton effects contribute to higher derivative terms in the effective action, as was shown for heterotic strings on T^6 in [4], or for type II compactifications on $K3 \times T^2$ in [5]. For a nice review, see [6]. The case of $N = 2$, corresponding to type II strings on CY-threefolds, is most interesting, as now the fivebrane instantons contribute to the low-energy effective action [7]. The aim of this paper is to compute some of these instanton effects explicitly in the semiclassical and supergravity approximation. A similar study for heterotic strings on CY-threefolds, with $N = 1$ supersymmetry in four dimensions, was initiated in [8]. Computations of nonperturbative superpotentials due to fivebrane instantons were given in [9].

In this paper, we thus focus on Neveu-Schwarz fivebrane instantons in type IIA string theory compactified on a CY manifold. In the absence of internal fluxes, a generic such compactification with Hodge numbers $h_{1,1}$ and $h_{1,2}$ has as its low-energy effective action four-dimensional $N = 2$ supergravity coupled to $h_{1,1}$ vector multiplets, $h_{1,2}$ hypermultiplets, and one tensor multiplet that contains the dilaton [10]. The simplest situation is therefore to freeze all the vector multiplets (they play no significant role in the present discussion since the other matter multiplets are neutral in the absence of fluxes) and to set $h_{1,2} = 0$. For a more general discussion on the geometry of vector and hypermultiplet moduli spaces, we refer to the review [11] and references therein.

Calabi-Yau manifolds with no complex structure moduli ($h_{1,2} = 0$) are called rigid. Truncating the vector multiplets, the resulting four-dimensional low-energy effective action is that of a tensor multiplet coupled to $N = 2$ supergravity, and the bosonic part of the

Lagrangian at string tree-level is given by[1]

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\mathrm{T}} = &-R - \frac{1}{2}\,\partial^\mu\phi\,\partial_\mu\phi + \frac{1}{2}\,\mathrm{e}^{2\phi}H^\mu H_\mu \\
&- \frac{1}{4}\,F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\,\mathrm{e}^{-\phi}\big(\partial^\mu\chi\,\partial_\mu\chi + \partial^\mu\varphi\,\partial_\mu\varphi\big) - \frac{1}{2}\,H^\mu\big(\chi\partial_\mu\varphi - \varphi\partial_\mu\chi\big)\,,
\end{aligned} \tag{1.2}
$$

where $H^\mu = \frac{1}{6}\,\varepsilon^{\mu\nu\rho\sigma}H_{\nu\rho\sigma}$ is the dual NS 2-form field strength. The first line comes from the NS sector in ten dimensions, and ϕ together with H^μ forms an $N = 1$ tensor multiplet that is also present in heterotic compactifications. It is then not surprising that the NS5-brane solution (1.1) straightforwardly descends to a finite action solution of the Euclidean tensor multiplet Lagrangian, with RR fields set to zero [8, 7, 12]. The second line descends from the RR sector. In particular, the graviphoton with field strength $F_{\mu\nu}$ descends from the ten-dimensional RR 1-form, and φ and χ can be combined into a complex scalar C that descends from the holomorphic components of the RR 3-form with (complex) indices along the holomorphic 3-form of the CY. Notice the presence of constant shift symmetries on both χ and φ. Together with a rotation on χ and φ they form a three-dimensional subgroup of symmetries. The presence of the RR scalars then opens up the possibility to construct more general instanton solutions for which the RR sector becomes nontrivial. We will extensively discuss what kind of RR backgrounds the NS5-brane instanton can support, and how the correlation functions depend on it. We should stress that these instantons are distinct from membrane instantons, arising from wrapping Euclidean D2-branes around three-cycles in the CY [7]. Membrane instantons have a different dependence on the string coupling constant, and have different charges. Their study is beyond the scope of the present paper.

The tensor multiplet Lagrangian (1.2) is dual to the universal hypermultiplet. This can be seen by dualizing the 2-form into an axionic pseudoscalar field σ, after which one obtains (modulo a surface term)

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\mathrm{UH}} = &-R - \frac{1}{4}\,F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\,\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}\,\mathrm{e}^{-\phi}\big(\partial^\mu\chi\,\partial_\mu\chi + \partial^\mu\varphi\,\partial_\mu\varphi\big) \\
&- \frac{1}{2}\,\mathrm{e}^{-2\phi}\big(\partial_\mu\sigma + \chi\partial_\mu\varphi\big)^2\,.
\end{aligned} \tag{1.3}
$$

The four scalars define the classical universal hypermultiplet at string tree-level, a non-linear sigma model with a quaternion-Kähler target space $\mathrm{SU}(1,2)/\mathrm{U}(2)$ [13]. This target space has an $\mathrm{SU}(1,2)$ group of isometries, with a three-dimensional Heisenberg subalgebra that generates the following shifts on the fields,

$$
\phi \to \phi\,, \qquad \chi \to \chi + \gamma\,, \qquad \varphi \to \varphi + \beta\,, \qquad \sigma \to \sigma - \alpha - \gamma\varphi\,, \tag{1.4}
$$

---

[1]Throughout this paper, we work in units in which Newton's constant $\kappa^{-2} = 2$, except in section 4, where we compare rigid and local $N = 2$ scalar-tensor couplings.

where α, β, γ are real (finite) parameters. A fourth symmetry is again a rotation on φ and χ, which is now accompanied by a compensating transformation on σ. The remaining four isometries involve non-trivial transformations on the dilaton, and hence will change the string coupling constant.

Quantum corrections, both perturbative and nonperturbative, will break some of the isometries and alter the classical moduli space of the universal hypermultiplet, while keeping the quaternion-Kähler property intact, as required by supersymmetry [14]. At the perturbative level, the authors of [15] reconsidered the analysis of [16, 17], and found a non-trivial one-loop correction that modifies the low-energy tensor multiplet Lagrangian (1.2), or after dualization, the universal hypermultiplet Lagrangian (1.3). More recently, this one-loop correction was written and analyzed in the language of projective superspace in [18], using the tools developed in [19]. At the nonperturbative level, not much is known explicitly at present. Some earlier references on this topic are [7, 20, 21, 12, 22, 23]. The problem of finding the quantum corrections to the hypermultiplet moduli space metric is somewhat similar to determining the hyperkähler metric on the Coulomb branch of three-dimensional gauge theories with eight supercharges [24, 25]. This analogy was used in [18] to conjecture a natural candidate metric on the universal hypermultiplet moduli space induced by five-brane instantons. Given the metric, one can then investigate the properties of the scalar potential that arises after gauging the remaining unbroken isometries. An interesting application, based on a proposal for the membrane instanton corrections to the universal hypermultiplet moduli space metric [23], was recently given in the context of finding de Sitter vacua from $N = 2$ gauged supergravity [26].

In this paper, elaborating on our earlier work in [27, 28], we take the first steps in computing the universal hypermultiplet moduli space metric explicitly by using semiclassical instanton calculations. As is familiar from Seiberg-Witten theory, and its three-dimensional version [24, 25], one has to be careful in matching the coordinates on the moduli space with the fields from the supergravity (or string) theory. In fact, using semiclassical techniques, we can only determine this relation at weak coupling, and one can think of the asymptotic value of the dilaton (the string coupling constant) as a radial coordinate on the moduli space. Away from the semiclassical regime, or equivalently, the asymptotic region of the moduli space, the relation between the fields and coordinates is expected to be more complicated, and using our approach, we have no access to this regime. Therefore, in this paper, we have to content ourselves with computing only the first non-vanishing, but leading exponential correction in the full moduli space metric. Subleading corrections have to be computed using other methods, or can perhaps be fixed by the (super) symmetry constraints (like for instance the quaternionic geometry) and the regularity assumptions of the moduli space. This paper is organized as follows: In section 2 we introduce the double-tensor multiplet

dual to the universal hypermultiplet. Section 3 contains a review of the fivebrane instanton and anti-instanton solutions present in the double-tensor multiplet, and we discuss their properties. Section 4 deals with the generic form of the effective action for scalar-tensor interactions, and we discuss the Euclidean supersymmetry rules which we apply to the universal double-tensor multiplet in section 5. In section 6, we show explicitly that the fivebrane instanton preserves half of the (Euclidean) supersymmetry, and we use the broken supersymmetries to determine the fermionic zero modes in section 7, both for instantons and anti-instantons. A crucial ingredient of any instanton calculation is finding the instanton measure on the space of bosonic and fermionic collective coordinates. For single-centered instantons, we determine this measure in section 8. The one-loop determinant is left unspecified, calculating it is beyond the scope of this paper. All this is preparatory material, and in section 9 we finally compute the instanton induced correlation functions, from which we determine the asymptotic corrections to the universal hypermultiplet moduli space in section 10. Not surprisingly, since the metric and other quantities for hypermultiplets are not holomorphic, contributions are found from both instantons and anti-instantons. We further discuss the structure of the isometry group, and the breaking of isometries to discrete subgroups, due to fivebrane instantons.

We end this paper with some conclusions and remarks for further investigation. To make this paper as readable as possible, we have included several appendices with additional information, conventions and technical details.

2 The universal double-tensor multiplet

Instead of dualizing the tensor multiplet Lagrangian (1.2) to a hypermultiplet, we can also use the shift symmetry of one of the RR scalars, say φ, to dualize to a double-tensor multiplet. This can only be done if the shift symmetry survives in the full quantum theory. In the presence of only NS5-brane instantons, this is indeed the case. We will come back to the discussion of broken isometries below. After dualization[2], the resulting tree level double-tensor multiplet Lagrangian reads [27, 29]

$$e^{-1}\mathcal{L}_{\mathrm{DT}} = -R - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\,\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}\,\mathrm{e}^{-\phi}\,\partial^\mu\chi\,\partial_\mu\chi + \frac{1}{2}\,M^{IJ}H_I^\mu H_{\mu J}\;, \qquad (2.1)$$

where the H_I are a pair of 3-form field strengths, $H_I^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma I}$, and

$$M^{IJ} = \mathrm{e}^\phi \begin{pmatrix} 1 & -\chi \\ -\chi & \mathrm{e}^\phi + \chi^2 \end{pmatrix}\;. \qquad (2.2)$$

---

[2]We are dualizing here at the classical level. Possible quantum corrections coming from the path integral measure are not taken into account.

The two scalars ϕ and χ parameterize the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{O}(2)$. More on the geometry of the target space before and after dualization can be found in appendix B. The presence of the tensors breaks the $\mathrm{SL}(2,\mathbb{R})$ symmetries to a two-dimensional subgroup generated by rescalings of the tensors and of $\tau \equiv \chi + 2\mathrm{i}\,\mathrm{e}^{\phi/2}$, and by the generator that acts as a shift on χ and transforms the tensors linearly into each other [27, 28],

$$\chi \to \chi + \gamma\ , \qquad B_1 \to B_1 + \gamma B_2\ , \tag{2.3}$$

with ϕ and B_2 invariant. This symmetry played a crucial role in determining the perturbative one-loop correction to the universal hypermultiplet. We will not discuss this perturbative correction any further in this paper, but refer to [15, 18].

In [27, 28], Bogomol'nyi equations were derived and solved for the double-tensor multiplet action (2.1). The solutions were shown to describe fivebrane and membrane-like instantons, and in this paper we focus on the fivebrane solutions. The semiclassical and supergravity description of these instantons is most natural in the Euclidean continuation of the double-tensor multiplet action. As explained in [27], this has the advantage over the Euclidean universal hypermultiplet or tensor multiplet in that the Euclidean action is semi-positive definite. Hence, a saddle-point approximation can be justified and a Bogomol'nyi bound can be derived. Both the tensor- and hypermultiplet actions contain pseudoscalars which have negative kinetic energy after analytic continuation. In such a formulation, the semiclassical approximation is hard to justify, and one has to add surface terms that guarantee the stability of the Euclidean hypermultiplet action. Dualizing the pseudoscalars into tensors leads to a formulation with a manifestly positive semi-definite action, and precisely produces the surface terms needed to make the pseudoscalar action bounded. This subtlety is similar to the D-instanton in type IIB supergravity in ten dimensions, where the axion is best dualized to a 9-form field strength [30]. See also [31] for related issues. For these reasons, we set up our instanton calculation in the double-tensor multiplet Lagrangian.

Perturbatively, the double-tensor multiplet guarantees $\mathrm{U}(1) \times \mathrm{U}(1)$ isometries in the dual hypermultiplet description. Nonperturbatively, however, the duality also involves the constant modes of the dual scalars φ and σ by means of theta-angle-like terms. These are surface terms which have to be added to the double-tensor multiplet Lagrangian, and which are non-vanishing in the presence of instantons and anti-instantons. In Euclidean space, they can be written as integrals over 3-spheres at infinity,

$$S_{\mathrm{surf}}^{E} = -\mathrm{i}\Big(\varphi \int_{S^3_\infty} H_1 + \sigma \int_{S^3_\infty} H_2\Big) = -\mathrm{i}\varphi Q_1 - \mathrm{i}\sigma Q_2\ . \tag{2.4}$$

Here, Q_1 and Q_2 are related to the membrane- (which we do not discuss in this paper) and fivebrane-instanton charges associated with the two tensors H_1 and H_2, and φ and σ are some parameters. In the dual hypermultiplet theory, they play the role of coordinates

on the moduli space. The dualization is done by promoting σ and φ to fields that serve as Lagrange multipliers enforcing the Bianchi identities on the tensors. Boundary terms such as (2.4) are also added in three-dimensional gauge theories in the Coulomb phase [32, 24, 25]. There, the effective theory can be described in terms of a vector that can be dualized into a scalar (the dual photon) along the same lines as described above. Clearly, depending on which charges are turned on (membrane or fivebrane), the shift symmetry in φ or σ (or some linear combination thereof) will be broken to a discrete subgroup. We will discuss the breaking of isometries in detail in section 10.

3 Fivebrane instantons

Our strategy is to perform a semiclassical instanton calculation in the double-tensor multiplet formulation, in the supergravity approximation. We should therefore first discuss the properties of the NS5-brane instanton, as a solution of the Euclidean equations of motion of the Lagrangian (2.1). As mentioned in the previous section, the equations that determine the instanton solutions were found in [27, 28] by deriving a Bogomol'nyi bound. The fivebrane instanton solution satisfies the Bogomol'nyi equation [27]

$$\begin{pmatrix} H_{\mu 1} \\ H_{\mu 2} \end{pmatrix} = \pm \partial_\mu \begin{pmatrix} \mathrm{e}^{-\phi}\chi \\ \mathrm{e}^{-\phi} \end{pmatrix} \ , \tag{3.1}$$

where the plus and minus signs refer to instantons or anti-instantons, respectively. It will often be useful to change basis and define $\hat{H}_1 = H_1 - \chi H_2$, which satisfies the Bogomol'nyi equation $\hat{H}_{\mu 1} = \pm \mathrm{e}^{-\phi}\partial_\mu\chi$. Such field configurations have vanishing energy-momentum tensors, so they must live in Ricci-flat spaces. In this paper we shall consider only flat space with metric $g_{\mu\nu} = \delta_{\mu\nu}$. Furthermore, we choose a vanishing graviphoton $F_{\mu\nu}$, and therefore we only focus on the scalar-tensor sector. The second equation in (3.1) comes from the NS sector and specifies the NS5-brane instanton. The first equation determines the RR sector and characterizes the RR background in which the fivebrane instanton lives. We now proceed to solve these equations.

The closure of the 3-form field strengths implies Laplace-like equations for the scalars. There are two approaches to proceed and solve these equations, which should be equivalent. The first one is to find solutions on the whole $\mathbb{R}^4$. This can only be done if appropriate source terms are added to the equations of motion. The other approach, which we will follow here, is to excise points $\{x_i\}$ from $\mathbb{R}^4$, the locations of the instantons. On such a space, we find the multi-centered solutions in terms of two harmonic functions,

$$\mathrm{e}^{-\phi} = \mathrm{e}^{-\phi_\infty} + \sum_i \frac{|Q_{2i}|}{4\pi^2\,(x-x_i)^2} \ , \qquad \mathrm{e}^{-\phi}\chi = \mathrm{e}^{-\phi_\infty}\chi_\infty + \sum_i \frac{Q_{1i}}{4\pi^2\,(x-x_i)^2} \ , \tag{3.2}$$

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where $Q_{1i}, Q_{2i}, \chi_\infty$, and ϕ_∞ are integration constants; the latter two determine the asymptotic values of the fields at infinity. The absolute values of Q_{2i} appear to make $e^{-\phi}$ positive everywhere in space. We identify the string coupling constant via $g_s = e^{-\phi_\infty/2}$. Furthermore, two charges are defined by integrating the tensor field strengths $H_{\mu\nu\rho\,I} = -\varepsilon_{\mu\nu\rho\sigma} H_I^\sigma$ over 3-spheres at infinity,

$$Q_I = \int_{S^3_\infty} H_I , \quad I = 1, 2 . \tag{3.3}$$

They are related to the constants appearing in the scalar fields through the field equation (3.1). Using $** = -1$ on a 3-form in four Euclidean dimensions, we find

$$Q_2 = \mp \sum_i |Q_{2i}| , \qquad Q_1 = \mp \sum_i Q_{1i} . \tag{3.4}$$

This implies that for instantons, Q_2 should be taken negative, whereas for anti-instantons, Q_2 must be positive[3]. Note that there is no restriction on the signs of the Q_{1i}.

The (anti-) instanton action for the fivebrane was found to be the following integral over the boundary of Euclidean spacetime [27]

$$S_{\rm cl} = \pm \int_{\partial \mathcal{M}} \left[\chi H_1 - \left(e^\phi + \tfrac{1}{2}\chi^2 \right) H_2 \right] , \tag{3.5}$$

where the boundary $\partial \mathcal{M}$ consists of the disjoint union of a sphere at infinity and infinitesimal spheres around the excised points x_i. The value of the action was computed and interpreted for general multicentered solutions in [28]. It is finite only if the values of χ at the excised points x_i are finite. We have

$$\chi_i \equiv \lim_{x \to x_i} \chi(x) = \frac{Q_{1i}}{|Q_{2i}|} , \tag{3.6}$$

which is finite whenever $Q_{2i} \neq 0$ for nonvanishing Q_{1i}. The finiteness condition can be rewritten as the vanishing of the charges

$$\hat{Q}_{1i} \equiv Q_{1i} - \chi_i |Q_{2i}| = 0 . \tag{3.7}$$

Plugging the solution into the action, we find

$$S_{\rm cl} = \frac{|Q_2|}{g_s^2} + \frac{1}{2} \sum_i |Q_{2i}| \left(\chi_\infty - \chi_i \right)^2 . \tag{3.8}$$

The $1/g_s^2$ dependence is consistent with the string theory expectations for a wrapped NS fivebrane around the entire CY [7].

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[3]Charges Q'_{Ii} can also be defined as the integrals of the field strengths over a 3-sphere around x_i. For anti-instantons they coincide with Q_{1i} and $|Q_{2i}|$, but for instantons, there are minus signs. To avoid heavy notation, we will never write Q'_{Ii}.

For a single-centered solution, finiteness of the action requires

$$\hat{Q}_1 \equiv Q_1 - \chi_0 Q_2 = 0 \;, \tag{3.9}$$

for both instantons and anti-instantons. Here, χ_0 is the value of the RR scalar at the location x_0 of the (anti-) instanton. Notice that, in contrast to the dilaton, the RR scalar remains finite at the origin. This regularity implies that the field equation for χ does not need a source. One should therefore think of χ_0 as some constant RR background flux in which the NS5-brane instanton with charge Q_2 lives. By means of (3.7), the quantization of the charges implies a quantization condition on the value of χ_0. In [28], $\hat{Q}_1$ was interpreted as a membrane instanton charge, because the corresponding instanton action is linear (instead of quadratic) in the inverse string coupling constant, in accordance with [7]. Therefore our solutions correspond to purely fivebrane instantons, with the addition of some RR flux χ_0, but with zero membrane charge. Notice that the membrane charge $\hat{Q}_1$ is invariant under the symmetry (2.3). The action for the single-centered fivebrane instanton then becomes

$$S_{\text{cl}} = |Q_2| \left(\frac{1}{g_s^2} + \frac{1}{2} (\Delta \chi)^2 \right) \;, \tag{3.10}$$

where $\Delta\chi = \chi_\infty - \chi_0$. The second term is the consequence of turning on a nontrivial RR background. This has the net effect of raising the instanton action by an amount of $(\Delta\chi)^2$. It vanishes when the boundary values of χ at $r = 0$ and $r = \infty$ are equal, i.e. when $\chi_\infty = \chi_0$. Plugging this into the solution (3.2), one easily checks that χ is constant everywhere, $\chi = \chi_\infty$. This is the most trivial RR background, and the easiest in which we can do instanton calculations. We will however also consider nonconstant backgrounds, and study how the fivebrane instanton responds to it.

Notice that the value of the action is positive and the same for instantons and anti-instantons. To distinguish instantons from anti-instantons, we add the theta-angle-like terms discussed in the introduction in (2.4). Using (3.9), we can rewrite this as

$$S_{\text{surf}} = -\mathrm{i}\sigma Q_2 - \mathrm{i}\varphi Q_1 = \pm \mathrm{i}(\sigma + \chi_0 \, \varphi) |Q_2| - \mathrm{i}\varphi \hat{Q}_1 \;. \tag{3.11}$$

The total instanton action is then

$$S_{\text{inst}}^{\pm} = S_{\text{cl}} + S_{\text{surf}} \;, \tag{3.12}$$

and will contribute to any instanton-induced correlation function by exponentiation. After dualization to the universal hypermultiplet, this instanton action will contribute to the effective moduli space metric, for which ϕ, χ, φ and σ can be thought of as the coordinates. As a consequence, some of the isometries will be broken, as we discuss in section 10.

In section 6 we show that, although these solutions satisfy the Bogomol'nyi bound, not all of them preserve half of the supersymmetries. For that, there will be further restrictions

on the multi-centered instanton solutions, namely that all the χ_i are equal. This implies that the solution is characterized in terms of a single harmonic function. When all the χ_i are equal, (3.8) reduces to the action of a single-centered instanton. As argued in [28], the generic multi-centered solution is expected to be metastable and decays into a solution where all χ_i are equal. The value of the action is then lowered to (3.10), and the points x^i can be brought together to obtain a spherically symmetric solution, without further changing the value of the action.

The actual instanton calculation we perform in later sections of this paper will be based on the single-centered instanton. The multi-centered ones can be written as a multipole expansion, in which the dominant term contributing to the low-energy effective action is the single-centered one. Before we compute these instanton effects, we must first discuss the general form of low-energy effective actions, and the (Euclidean) supersymmetry transformation rules. The latter are needed to discuss the BPS properties of our instanton solutions.

4   Low energy effective actions for scalar-tensor multiplets

The $N = 2$ scalar-tensor system consists of n_T tensors $B_{\mu\nu I}$, $I = 1, \ldots, n_T$ and $4n - n_T$ scalars ϕ^A, together with $2n$ two-component spinors λ^a. Its self-interactions and its couplings to $N = 2$ supergravity in spacetimes with Lorentz signature have been derived in [29]. We will not repeat it here, but only discuss its analytic continuation to Euclidean signature. The case of interest for our applications corresponds to $n = 1$ and $n_T = 2$, but the discussion below is for arbitrary scalar-tensor systems. The Wick rotation can easily be done by using the prescription that under $x^0 \to -\mathrm{i}x^4$ we have

$$B_{0i} \to \mathrm{i}B_{4i} , \qquad B_{ij} \to B_{ij} , \qquad i = 1, 2, 3 . \tag{4.1}$$

More conventions on spinors and the Euclidean Clifford algebra are given in appendix A.

Rigid Supersymmetry

The Euclidean action can be found by doing the Wick rotation as described above. After extracting an overall minus sign, such that the action enters the path integral via e^{-S}, it is given by

$$\mathcal{L} = \frac{1}{2} M^{IJ} H_I^\mu H_{\mu J} + \frac{1}{2} \mathcal{G}_{AB} \, \partial^\mu \phi^A \, \partial_\mu \phi^B - \mathrm{i}A_A^I \, H_I^\mu \, \partial_\mu \phi^A + \frac{\mathrm{i}}{2} \, h_{a\bar{a}} \left(\lambda^a \sigma^\mu \overset{\leftrightarrow}{\mathcal{D}}_\mu \bar{\lambda}^{\bar{a}} \right)$$

$$+ \mathrm{i}H_{\mu I} M^{IJ} k_{Ja\bar{a}} \, \lambda^a \sigma^\mu \bar{\lambda}^{\bar{a}} - \frac{1}{4} V_{ab\,\bar{a}\bar{b}} \, \lambda^a \lambda^b \, \bar{\lambda}^{\bar{a}} \bar{\lambda}^{\bar{b}} , \tag{4.2}$$

for some functions $M^{IJ}(\phi)$, $\mathcal{G}_{AB}(\phi)$, etc. We denote $H_I^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma I}$, and the covariant derivative is given by

$$\mathcal{D}_\mu \lambda^a = \partial_\mu \lambda^a + \partial_\mu \phi^A \, \Gamma_A{}^a{}_b \, \lambda^b . \tag{4.3}$$

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The connection ensures covariance with respect to fermion frame reparametrizations $\lambda^a \to S^a{}_b(\phi)\lambda^b$.

In the above we have assumed that all the fields ϕ^A stay inert under the Wick rotation. This guarantees that, for a positive definite metric $\mathcal{G}_{AB}$, the kinetic terms for the scalars contribute positively to the action. Sometimes, when dealing with pseudoscalars, one could pick up a factor of i upon the Wick rotation such that the action contains negative kinetic terms. We assume that all such fields with negative kinetic energy can be dualized into tensors, and hence we have a positive kinetic energy for the tensors, with positive definite M^{IJ} (this assumption is satisfied for the universal hypermultiplet). One can indeed easily check that the dualization in Euclidean space changes the sign of the kinetic term. The conventions are such that the matrices $M^{IJ}, \mathcal{G}_{AB}$ and A^I_A are the same as in Minkowski space. Sign changes or factors of i upon Wick rotation are never absorbed into these quantities, but written explicitly.

In Lorentz signature, the complete supersymmetry transformation rules were given in [29]. For our purpose of calculating instanton effects, it suffices to know the linearized supersymmetry transformation rules. In Euclidean space, and for rigid supersymmetry, they can be obtained by using (4.1) and (A.1) in the Lorentzian transformation rules:

$$\delta_\epsilon \phi^A = \gamma^A_{ia}\,\epsilon^i \lambda^a + \bar\gamma^{iA}_{\bar a}\,\bar\epsilon_i\,\bar\lambda^{\bar a}$$

$$\delta_\epsilon B_{\mu\nu I} = 2\mathrm{i}\,g_{Iia}\,\epsilon^i \sigma_{\mu\nu}\lambda^a - 2\mathrm{i}\,\bar g^i_{I\bar a}\,\bar\epsilon_i\,\bar\sigma_{\mu\nu}\bar\lambda^{\bar a}$$

$$\delta_\epsilon \lambda^a = \left(\mathrm{i}\partial_\mu \phi^A\,W^{ai}_A + H_{\mu I} f^{Iai}\right)\sigma^\mu \bar\epsilon_i + \dots$$

$$\delta_\epsilon \bar\lambda^{\bar a} = \left(\mathrm{i}\partial_\mu \phi^A\,\bar W^{\bar a}_{Ai} + H_{\mu I}\bar f^{I\bar a}{}_i\right)\bar\sigma^\mu \epsilon^i + \dots\;, \tag{4.4}$$

for some functions γ^A_{ia}, g_{Iia} etc. The requirement of closure of the supersymmetry algebra and invariance of the action imposes constraints on and relations between the various quantities appearing in the action and supersymmetry transformation rules. With the parametrization given above, they are exactly the same as in Lorentz signature [29], and we repeat them in appendix C. The ellipsis stands for higher order terms in the fermions that do not play a role at the linearized level. In comparison with the Lorentzian case, the transformation rules for the bosonic fields are the same, whereas for the fermions a factor of i appears in the variation proportional to the tensor fields, as is consistent with the rules of the Wick rotation.

Local Supersymmetry

We now couple the system to Euclidean $N = 2$ supergravity. Pure $N = 2$ supergravity in Euclidean space, without any matter couplings, was constructed in [33]. The scalar-tensor self-interactions in Euclidean space were described above, and now we need its coupling to the $N = 2$ Poincaré supergravity multiplet, which contains a vielbein $e_\mu{}^m$, the graviphoton

A_μ, and two gravitinos ψ_μ^i, $\bar\psi_{\mu i}$, $i = 1, 2$. We follow the same Wick rotation rules as for rigid supersymmetry, replace the metric by a Euclidean metric, supplemented by

$$\psi_0^i \to \mathrm{i}\psi_4^i \,, \qquad \bar\psi_{0i} \to \mathrm{i}\bar\psi_{4i} \,, \qquad A_0 \to \mathrm{i}A_4 \,, \tag{4.5}$$

for the gravitino and graviphoton. Notice again that in Euclidean space ψ_μ^i and $\bar\psi_{\mu i}$ are no longer related by complex conjugation.

In Lorentz signature, the action and supersymmetry transformation rules were given in [29], and after Wick rotation we get

$$
\begin{aligned}
e^{-1}\mathcal{L} &= \frac{1}{2\kappa^2}\,R + \frac{1}{4}\,\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}\,\mathcal{G}_{AB}\,\hat{D}^\mu\phi^A\,\hat{D}_\mu\phi^B + \frac{1}{2}\,M^{IJ}\mathcal{H}_I^\mu\,\mathcal{H}_{\mu J} - \mathrm{i}A_A^I H_I^\mu\,\partial_\mu\phi^A \\
&\quad + \mathrm{i}\varepsilon^{\mu\nu\rho\sigma}(\mathcal{D}_\mu\psi_\nu^i\sigma_\rho\bar\psi_{\sigma i} + \psi_\sigma^i\sigma_\rho\mathcal{D}_\mu\bar\psi_{\nu i}) + \frac{\mathrm{i}}{2}\,h_{a\bar a}\,(\lambda^a\sigma^\mu\mathcal{D}_\mu\bar\lambda^{\bar a} - \mathcal{D}_\mu\lambda^a\sigma^\mu\bar\lambda^{\bar a}) \\
&\quad - \kappa\,\mathcal{G}_{AB}(\hat{D}_\mu\phi^A + \partial_\mu\phi^A)\,(\gamma_{ia}^B\,\lambda^a\sigma^{\mu\nu}\psi_\nu^i + \text{c.c.}) + \mathrm{i}\kappa M^{IJ}\mathcal{H}_I^\mu\,(g_{Jia}\,\psi_\mu^i\lambda^a + \text{c.c.}) \\
&\quad - \mathrm{i}\frac{\kappa}{2\sqrt{2}}\,(\tilde{\mathcal{F}}^{\mu\nu} + \tilde{F}^{\mu\nu})\,(\psi_\mu^i\psi_{\nu i} + \bar\psi_\mu^i\bar\psi_{\nu i}) + \frac{\mathrm{i}\kappa}{2\sqrt{2}}\,\mathcal{F}_{\mu\nu}\,(\mathcal{E}_{ab}\,\lambda^a\sigma^{\mu\nu}\lambda^b - \text{c.c.}) \\
&\quad + \mathrm{i}M^{IJ}k_{Ja\bar a}\,\lambda^a\sigma^\mu\bar\lambda^{\bar a}\big[\mathcal{H}_{\mu I} + \mathrm{i}\kappa\,(g_{Iib}\,\psi_\mu^i\lambda^b + \text{c.c.})\big] \\
&\quad + \kappa^2 M^{IJ}(g_{Iia}\,\psi_\mu^i\lambda^a + \text{c.c.})\,(g_{Jjb}\,\lambda^b\sigma^{\mu\nu}\psi_\nu^j + \text{c.c.}) \\
&\quad + \frac{\kappa^2}{8}\,(\mathcal{E}_{ac}\mathcal{E}_{bd}\,\lambda^a\lambda^b\,\lambda^c\lambda^d + \text{c.c.}) - \frac{1}{4}\,V_{ab\,\bar a\bar b}\,\lambda^a\lambda^b\,\bar\lambda^{\bar a}\bar\lambda^{\bar b} \,. \tag{4.6}
\end{aligned}
$$

By "c.c." above, and further on below, we mean the analytic continuation of the complex conjugated expressions in Lorentz signature. The covariant derivatives $\mathcal{D}_\mu$ of the fermions contain connections $\Gamma_A{}^a{}_b$ and $\Gamma_A{}^i{}_j$, just in the same way as in Minkowski space. We refer to [29] for more details about the connections, supercovariant derivatives and field strengths. Notice also the appearance of a new four-fermi term proportional to the tensor $\mathcal{E}_{ab}$ as defined in (C.7).

It is important to mention here that we are working in a 1.5 order formalism, in which the spin-connection is determined by its own field equation. Hence, it contains gravitinos as well as matter fermion bilinears. This formalism simplifies checking supersymmetry of the action, but hides certain quartic fermion terms, both in the gravitino and in the matter sector. One would therefore have to be careful in interpreting the calculation of instanton corrections to four-fermi correlators, which would be nonzero as a result of the zero mode counting. Fortunately, as we will see later on, there is a more convenient way that avoids this source of confusion.

The Euclidean linearized supersymmetry transformation rules are again given by (4.4), where all derivatives and field strengths are replaced by supercovariant ones, and [29]

$$\delta_\epsilon B_{\mu\nu I} = 2\mathrm{i}\,g_{Iia}\,\epsilon^i\sigma_{\mu\nu}\lambda^a - 4\kappa^{-1}\Omega_I{}^i{}_j\,\epsilon^j\sigma_{[\mu}\bar\psi_{\nu]i} + \text{c.c.} \,. \tag{4.7}$$

The transformations of the supergravity multiplet are given by

$$\delta_\epsilon e_\mu{}^m = i\kappa \left(\epsilon^i \sigma^m \bar{\psi}_{\mu i} - \psi^i_\mu \sigma^m \bar{\epsilon}_i \right)$$

$$\delta_\epsilon A_\mu = i\sqrt{2} \left(\epsilon_i \psi^i_\mu + \bar{\epsilon}^i \bar{\psi}_{\mu i} \right)$$

$$\delta_\epsilon \psi^i_\mu = \kappa^{-1} \mathcal{D}_\mu \epsilon^i + \frac{1}{\sqrt{2}} \varepsilon^{ij} F^+_{\mu\nu} \sigma^\nu \bar{\epsilon}_j - i\kappa^{-1} H_{\mu I} \Gamma^{Ii}{}_j \epsilon^j + \dots$$

$$\delta_\epsilon \bar{\psi}_{\mu i} = \kappa^{-1} \mathcal{D}_\mu \bar{\epsilon}_i + \frac{1}{\sqrt{2}} \varepsilon_{ij} F^-_{\mu\nu} \bar{\sigma}^\nu \epsilon^j + i\kappa^{-1} H_{\mu I} \Gamma^{Ij}{}_i \bar{\epsilon}_j + \dots \ , \tag{4.8}$$

where in the last two lines we have denoted the (anti-) selfdual graviphoton field strengths by $F^\pm_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} \pm \tilde{F}_{\mu\nu})$, and we have dropped fermion bilinears. Just as for the rigid case, they can easily be reinserted.

The coefficient functions g_{Iia}, W^{ai}_A, etc. appearing in the above equations are related to hypermultiplet quantities in the same way as in the rigid case. Hence, they satisfy the same relations (C.1)–(C.8). Moreover, we have the relation

$$\Gamma^{Ii}{}_j = M^{IJ} \Omega_J{}^i{}_j \ , \tag{4.9}$$

between the coefficients which appear in the supersymmetry transformations of the gravitinos and tensors, respectively.

5 The supersymmetric double-tensor multiplet

The purpose of the previous section was twofold: to give the generic form of the low-energy effective action, and to discuss the supersymmetry rules that will be needed to study the BPS properties of the fivebrane instantons that appear in the (classical) double-tensor multiplet action (2.1). This multiplet and its action are a specific example of the general case given in (4.6), for which[4]

$$\mathcal{G}_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\phi} \end{pmatrix} , \qquad M^{IJ} = e^\phi \begin{pmatrix} 1 & -\chi \\ -\chi & e^\phi + \chi^2 \end{pmatrix} , \qquad A^I_A = 0 . \tag{5.1}$$

Strictly speaking, instead of $A^I_A = 0$ we could allow for a nonvanishing connection with trivial field strength $F_{AB}{}^I = 2\partial_{[A} A^I_{B]} = 0$. Such connections are pure gauge and lead to total derivatives in the action. In perturbation theory, they can be dropped, but nonperturbatively they can be nonvanishing and lead to imaginary theta-angle-like terms. We have discussed and included such terms seperately in (2.4) and (3.11), so it suffices to set $A^I_A = 0$. The complete double-tensor multiplet, including all other coefficient functions

---

[4]From now on we set again $\kappa^{-2} = 2$ and rescale the supersymmetry parameters ϵ^i, $\bar{\epsilon}_i$ by a factor of $\sqrt{2}$.

that determine the action and supersymmetry transformations, was written down in [27], and is summarized in appendix D.

Using these results, we can now give the linearized (in the fermions) Euclidean supersymmetry transformations of the fermions. They can be written as

$$\delta_\epsilon \lambda^a = \mathrm{i}\sqrt{2}\, E_\mu^{ai}\, \sigma^\mu \bar{\epsilon}_i \,, \qquad \delta_\epsilon \bar{\psi}_{\mu i} = 2\bar{D}_{\mu i}{}^j\, \bar{\epsilon}_j + \varepsilon_{ij} F^-_{\mu\nu}\, \bar{\sigma}^\nu \epsilon^j$$

$$\delta_\epsilon \bar{\lambda}^{\bar{a}} = \mathrm{i}\sqrt{2}\, \bar{E}^{\bar{a}}_{\mu i}\, \bar{\sigma}^\mu \epsilon^i \,, \qquad \delta_\epsilon \psi_\mu^i = 2D_\mu{}^i{}_j\, \epsilon^j + \varepsilon^{ij} F^+_{\mu\nu}\, \sigma^\nu \bar{\epsilon}_j \,, \tag{5.2}$$

where we have introduced

$$E_\mu^{ai} = \partial_\mu \phi^A W_A^{ai} - \mathrm{i}H_{\mu I} f^{Iai} \,, \qquad \bar{D}_{\mu i}{}^j = \delta_i^j \nabla_\mu - \partial_\mu \phi^A \Gamma_A{}^j{}_i + \mathrm{i}H_{\mu I}\Gamma^{Ij}{}_i$$

$$\bar{E}^{\bar{a}}_{\mu i} = \partial_\mu \phi^A \bar{W}_{Ai}^{\bar{a}} - \mathrm{i}H_{\mu I}\bar{f}^{I\bar{a}}{}_i \,, \qquad D_\mu{}^i{}_j = \delta_j^i \nabla_\mu + \partial_\mu \phi^A \Gamma_A{}^i{}_j - \mathrm{i}H_{\mu I}\Gamma^{Ii}{}_j \,, \tag{5.3}$$

with ∇_μ the Lorentz-covariant derivative. We will find useful the observation that $\bar{E}_\mu$ and D_μ are related to their counterparts E_μ and $\bar{D}_\mu$ according to[5]

$$\bar{E}^{\bar{a}}_{\mu j} = -h^{\bar{a}a}\mathcal{E}_{ab} E_\mu^{bl}\, \varepsilon_{lj} \,, \qquad D_\mu{}^i{}_j = \varepsilon^{ik}\bar{D}_{\mu k}{}^l\, \varepsilon_{lj} \,. \tag{5.4}$$

The first identity is due to the relation (C.8), while the second is a consequence of SU(2)-covariant constancy of ε_{ij}.

More explicitly, we have, at the linearized level,

$$\delta_\epsilon \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} = \mathrm{i}\begin{pmatrix} \mathrm{e}^{-\phi/2}\partial_\mu\chi - \mathrm{e}^{\phi/2}\hat{H}_{\mu 1} & \partial_\nu\phi + \mathrm{e}^{\phi}H_{\nu 2} \\ -\partial_\mu\phi + \mathrm{e}^{\phi}H_{\mu 2} & \mathrm{e}^{-\phi/2}\partial_\nu\chi + \mathrm{e}^{\phi/2}\hat{H}_{\nu 1} \end{pmatrix}\begin{pmatrix} \sigma^\mu\bar{\epsilon}_1 \\ \sigma^\nu\bar{\epsilon}_2 \end{pmatrix}$$

$$\delta_\epsilon \begin{pmatrix} \bar{\lambda}^1 \\ \bar{\lambda}^2 \end{pmatrix} = \mathrm{i}\begin{pmatrix} \mathrm{e}^{-\phi/2}\partial_\mu\chi + \mathrm{e}^{\phi/2}\hat{H}_{\mu 1} & \partial_\nu\phi - \mathrm{e}^{\phi}H_{\nu 2} \\ -\partial_\mu\phi - \mathrm{e}^{\phi}H_{\mu 2} & \mathrm{e}^{-\phi/2}\partial_\nu\chi - \mathrm{e}^{\phi/2}\hat{H}_{\nu 1} \end{pmatrix}\begin{pmatrix} \bar{\sigma}^\mu\epsilon^1 \\ \bar{\sigma}^\nu\epsilon^2 \end{pmatrix} \tag{5.5}$$

for the matter fermions, and

$$\delta_\epsilon \begin{pmatrix} \psi_\mu^1 \\ \psi_\mu^2 \end{pmatrix} = \begin{pmatrix} 2\nabla_\mu + \tfrac{1}{2}\mathrm{e}^{\phi}H_{\mu 2} & -\mathrm{e}^{-\phi/2}\partial_\mu\chi + \mathrm{e}^{\phi/2}\hat{H}_{\mu 1} \\ \mathrm{e}^{-\phi/2}\partial_\mu\chi + \mathrm{e}^{\phi/2}\hat{H}_{\mu 1} & 2\nabla_\mu - \tfrac{1}{2}\mathrm{e}^{\phi}H_{\mu 2} \end{pmatrix}\begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix} + \dots$$

$$\delta_\epsilon \begin{pmatrix} \bar{\psi}_{\mu 1} \\ \bar{\psi}_{\mu 2} \end{pmatrix} = \begin{pmatrix} 2\nabla_\mu - \tfrac{1}{2}\mathrm{e}^{\phi}H_{\mu 2} & -\mathrm{e}^{-\phi/2}\partial_\mu\chi - \mathrm{e}^{\phi/2}\hat{H}_{\mu 1} \\ \mathrm{e}^{-\phi/2}\partial_\mu\chi - \mathrm{e}^{\phi/2}\hat{H}_{\mu 1} & 2\nabla_\mu + \tfrac{1}{2}\mathrm{e}^{\phi}H_{\mu 2} \end{pmatrix}\begin{pmatrix} \bar{\epsilon}_1 \\ \bar{\epsilon}_2 \end{pmatrix} + \dots \tag{5.6}$$

for the gravitinos, where we have omitted the graviphoton terms.

We end this section by giving the fermionic equations of motion, at the linearized level. For the hyperinos we find

$$\mathrm{i}\sigma^\mu \mathcal{D}_\mu \bar{\lambda}^{\bar{a}} + H_{\mu I}\bar{\Gamma}^{I\bar{a}}{}_{\bar{b}}\,\sigma^\mu\bar{\lambda}^{\bar{b}} + \frac{\mathrm{i}}{2}h^{\bar{a}a}\mathcal{E}_{ab}F^+_{\mu\nu}\sigma^{\mu\nu}\lambda^b = -\frac{1}{\sqrt{2}}\sigma^\nu\bar{E}^{\bar{a}}_{\mu i}\bar{\sigma}^\mu\psi_\nu^i \tag{5.7}$$

---

[5]Note that in the second identity the covariant derivatives in D_μ and $\bar{D}_\mu$ are in the same representation of Spin(4), whereas in (5.2) they are not.

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$$\mathrm{i}\bar{\sigma}^\mu \mathcal{D}_\mu \lambda^a + H_{\mu I}\,\Gamma^{Ia}{}_b\,\bar{\sigma}^\mu \lambda^b - \frac{\mathrm{i}}{2}\,h^{\bar{a}a}\bar{\mathcal{E}}_{\bar{a}\bar{b}}\,F^-_{\mu\nu}\,\bar{\sigma}^{\mu\nu}\bar{\lambda}^{\bar{b}} = -\frac{1}{\sqrt{2}}\,\bar{\sigma}^\nu E^{ai}_\mu \sigma^\mu \bar{\psi}_{\nu\,i}\ . \tag{5.8}$$

What makes these different from the usual Dirac-like equation is the presence of the mixing term with the (anti-) selfdual graviphoton field strenth and the inhomogeneous gravitino term originating from its coupling to the rigid supersymmetry current of the double-tensor multiplet. This will become important in the discussion of the fermionic zero modes. The gravitino field equations read

$$\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\sigma_\rho \mathcal{D}_\sigma \bar{\psi}_{\nu\,i} - \varepsilon^{\mu\nu\rho\sigma}H_{\sigma I}\Gamma^{Ij}{}_i\,\sigma_\rho \bar{\psi}_{\nu j} - \mathrm{i}F^{\mu\nu-}\psi_{\nu\,i} = \frac{1}{2\sqrt{2}}\,h_{a\bar{a}}\bar{E}^{\bar{a}}_{\nu\,i}\,\sigma^\nu \bar{\sigma}^\mu \lambda^a \tag{5.9}$$

$$\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\bar{\sigma}_\rho \mathcal{D}_\sigma \psi^i_\nu + \varepsilon^{\mu\nu\rho\sigma}H_{\sigma I}\Gamma^{Ii}{}_j\,\bar{\sigma}_\rho \psi^j_\nu + \mathrm{i}F^{\mu\nu+}\bar{\psi}^i_\nu = -\frac{1}{2\sqrt{2}}\,h_{a\bar{a}}E^{ai}_\nu\,\bar{\sigma}^\nu \sigma^\mu \bar{\lambda}^{\bar{a}}\ . \tag{5.10}$$

Notice that one can combine the first two terms on the left-hand side into the operator $D_\mu{}^i{}_j$, as defined in (5.3).

The fermionic field equations will be important for finding the fermionic zero modes. We return to this in section 7.

6   Unbroken supersymmetries

We now determine the supersymmetries left unbroken by our instanton solutions. These are generated by Killing spinors ϵ^i, $\bar{\epsilon}_i$ which give vanishing supersymmetry transformations (5.2) of the fermions in the bosonic instanton background. We shall concentrate on the parameters $\bar{\epsilon}_i$ in the following, the ϵ^i can be obtained from these as explained below. We begin by writing the λ^a transformations as

$$\delta_\epsilon \lambda^1 = -2\mathrm{i}\,\sigma^\mu (\nabla_\mu - \tfrac{1}{2}\partial_\mu \phi - \tfrac{1}{4}\mathrm{e}^\phi H_{\mu 2})\bar{\epsilon}_2 + \mathrm{i}\sigma^\mu \delta_\epsilon \bar{\psi}_{\mu 2}$$
$$\delta_\epsilon \lambda^2 = 2\mathrm{i}\,\sigma^\mu (\nabla_\mu - \tfrac{1}{2}\partial_\mu \phi + \tfrac{1}{4}\mathrm{e}^\phi H_{\mu 2})\bar{\epsilon}_1 - \mathrm{i}\sigma^\mu \delta_\epsilon \bar{\psi}_{\mu 1}\ , \tag{6.1}$$

which holds identically for any purely bosonic field configuration with $F^-_{\mu\nu} = 0$. Requiring the transformations of the fermions to vanish imposes the necessary conditions

$$\sigma^\mu (\nabla_\mu - \tfrac{1}{2}\partial_\mu \phi + \tfrac{1}{4}\mathrm{e}^\phi H_{\mu 2})\,\bar{\epsilon}_1 = 0$$
$$\sigma^\mu (\nabla_\mu - \tfrac{1}{2}\partial_\mu \phi - \tfrac{1}{4}\mathrm{e}^\phi H_{\mu 2})\,\bar{\epsilon}_2 = 0\ . \tag{6.2}$$

They simplify upon using the Bogomol'nyi condition. Eq. (3.1) implies that $\delta_\epsilon \lambda^1 = \delta_\epsilon \bar{\lambda}^2 = 0$ identically for instantons ($\delta_\epsilon \lambda^2 = \delta_\epsilon \bar{\lambda}^1 = 0$ for anti-instantons). Accordingly, the solutions $\bar{\epsilon}_i$ to (6.2) will also solve $\sigma^\mu \delta_\epsilon \bar{\psi}_{\mu 2} = 0$. Using the Bogomol'nyi condition for $H_{\mu 2}$ in (6.2), we can write the $\bar{\epsilon}_i$ as

$$\bar{\epsilon}_1 = \mathrm{e}^{(2\pm 1)\phi/4}\,\bar{\eta}_1\ , \qquad \bar{\epsilon}_2 = \mathrm{e}^{(2\mp 1)\phi/4}\,\bar{\eta}_2\ , \tag{6.3}$$

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where the spinors $\bar{\eta}_i$ have to satisfy $\nabla\!\!\!\!/\,\bar{\eta}_i = 0$. We consider only flat space as a background in this paper, and zero modes of the operator $\partial\!\!\!/$ will be discussed in detail in the next section. We shall find there that they are of the form $\bar{\eta} = \text{const} + \partial_\mu h\,\bar{\sigma}^\mu\xi$, where $h(x)$ is a harmonic function and ξ a constant spinor. For instantons the condition

$$\tfrac{1}{2}\delta_\epsilon\bar{\psi}_{\mu 2} = (\partial_\mu - \tfrac{1}{4}\partial_\mu\phi)\,\bar{\epsilon}_2 = \mathrm{e}^{\phi/4}\,\partial_\mu\bar{\eta}_2 = 0 \tag{6.4}$$

leaves only $\bar{\eta}_2 = \text{const}$. It remains to consider the transformation of $\bar{\psi}_{\mu 1}$; using the Bogomol'nyi condition for both tensors and the above expressions for $\bar{\epsilon}_i$ yields

$$\tfrac{1}{2}\delta_\epsilon\bar{\psi}_{\mu 1} = (\partial_\mu + \tfrac{1}{4}\partial_\mu\phi)\,\bar{\epsilon}_1 - \mathrm{e}^{-\phi/2}\partial_\mu\chi\,\bar{\epsilon}_2 = \mathrm{e}^{-\phi/4}\,\partial_\mu(\mathrm{e}^{\phi}\bar{\eta}_1 - \chi\bar{\eta}_2) = 0\ . \tag{6.5}$$

Using the fact that $\bar{\eta}_2 = \text{const}$ and $\bar{\eta}_1 = \text{const} + \partial_\mu h\,\bar{\sigma}^\mu\xi$ leads to two possibilities. For $\bar{\eta}_1 = \text{const}$ it follows that $\bar{\eta}_1 = \rho\,\bar{\eta}_2$ and

$$\chi = \rho\,\mathrm{e}^{\phi} + \chi_0\ , \tag{6.6}$$

where χ_0, ρ are constants. Using the notation introduced in section 3, we can rewrite $\rho = g_s^2\Delta\chi$. (6.6) constrains the instanton configuration beyond the restrictions imposed by the Bogomol'nyi condition. As explained in section 3, it requires that all χ_i are equal. Note that this relation between χ and ϕ follows automatically if one imposes spherical symmetry, since then the harmonic function $\mathrm{e}^{-\phi}\chi$ must depend linearly on the harmonic function $\mathrm{e}^{-\phi}$ [27].

For $\bar{\eta}_1 = \partial_\mu h\,\bar{\sigma}^\mu\xi$, we did not find any non-trivial solutions.

The same analysis can be done for anti-instantons. In either case $\bar{\eta}_1$ turns out to be proportional to $\bar{\eta}_2$, so we conclude that the unbroken supersymmetries $\bar{\epsilon}_i$ are given in terms of one constant spinor $\bar{\eta}$ as $\bar{\epsilon}_i(x) = u_i(x)\,\bar{\eta}$, with functions

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathrm{e}^{\phi/2}\begin{pmatrix} \rho^{(1\pm 1)/2}\,\mathrm{e}^{\pm\phi/4} \\ \pm\rho^{(1\mp 1)/2}\,\mathrm{e}^{\mp\phi/4} \end{pmatrix}\ . \tag{6.7}$$

The Killing spinors of opposite chirality are then given by $\epsilon^i = \varepsilon^{ij}u_j\,\eta$, with η another arbitrary constant spinor. This immediately follows from the relations (5.4): if u_i are (spinless) zero modes of E_μ and $\bar{D}_\mu$, then $\varepsilon^{ij}u_j$ are zero modes of $\bar{E}_\mu$ and D_μ. We conclude that the fivebrane instanton in flat space, subject to the additional constraint (6.6), preserves one half of the supersymmetries[6].

---

[6]The trivial solution with both $\mathrm{e}^{-\phi}$ and χ constants of course preserves all the supersymmetries. Such solutions are not instantons but rather, they parametrize the vacua which the instantons interpolate between.

7 Fermionic zero modes and broken supersymmetries

In this section we determine the fermionic zero modes and show that all of them can be obtained by acting with the broken supersymmetries on the purely bosonic solution. We first concentrate on instantons; the case of anti-instantons is similar and will be summarized at the end of this section.

Spin 1/2 zero modes

The zero modes for the matter fermions are given by the solutions to the linearized hyperino equations of motion (5.7), which are coupled to the gravitinos. For the double-tensor multiplet we have $\Gamma_A{}^a{}_b = \Gamma^1{}^a{}_b = 0$; plugging in the only nonvanishing coefficient $\Gamma^2{}^a{}_b$ given in (D.6) and using the Bogomol'nyi condition (3.1) (with the upper sign) for $H_{\mu 2}$, the two hyperino equations become

$$
i\sigma^\mu \begin{pmatrix} \partial_\mu \bar\lambda^1 - \tfrac{3}{4}\partial_\mu\phi\, \bar\lambda^1 \\ \partial_\mu \bar\lambda^2 + \tfrac{3}{4}\partial_\mu\phi\, \bar\lambda^2 \end{pmatrix} = -\sigma^\nu\bar\sigma^\mu \begin{pmatrix} e^{-\phi/2}\partial_\mu\chi\, \psi^1_\nu + \partial_\mu\phi\, \psi^2_\nu \\ 0 \end{pmatrix} . \tag{7.1}
$$

Notice that the $\bar\lambda^2$ equation has no coupling to the gravitinos. Similarly for the unbarred hyperinos,

$$
i\bar\sigma^\mu \begin{pmatrix} \partial_\mu \lambda^1 + \tfrac{3}{4}\partial_\mu\phi\, \lambda^1 \\ \partial_\mu \lambda^2 - \tfrac{3}{4}\partial_\mu\phi\, \lambda^2 \end{pmatrix} = \bar\sigma^\nu\sigma^\mu \begin{pmatrix} 0 \\ \partial_\mu\phi\, \bar\psi_{\nu 1} - e^{-\phi/2}\partial_\mu\chi\, \bar\psi_{\nu 2} \end{pmatrix} , \tag{7.2}
$$

but now λ^1 decouples from the gravitinos.

It is useful to first study zero modes of the operator

$$
\slashed{D}_k \equiv \sigma^\mu(\partial_\mu - k\partial_\mu\phi) = e^{k\phi}\slashed{\partial}\, e^{-k\phi} , \qquad k \in \mathbb{R} . \tag{7.3}
$$

In the absence of gravitinos, this is precisely the relevant zero mode operator. Clearly, the zero modes of $\slashed{D}_k$ are in one-to-one correspondence with those of $\slashed{\partial}$. But whereas solutions $\bar\zeta$ to $\slashed{\partial}\bar\zeta = 0$ are not normalizable, the corresponding modes $\bar\lambda = e^{k\phi}\bar\zeta$ of $\slashed{D}_k$ can be normalizable for appropriate values of k. In flat Euclidean space, with the origin x_0 not excised, the only solution for $\bar\zeta$ is a constant spinor. However, when the origin is cut out, there is another non-trivial solution:

$$
\bar\zeta(x) = 2i\, \partial_\mu h(x)\, \bar\sigma^\mu\xi , \tag{7.4}
$$

where ξ is a constant spinor, h is a harmonic function, and the factor 2i is a choice of normalization. This is the only solution, as one can show by rewriting the two-component spinor as $\bar\zeta(x) = 2i\, f_\mu(x)\, \bar\sigma^\mu\xi$ for arbitrary real functions f_μ; the equation $\slashed{\partial}\bar\zeta = 0$ then

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imposes the conditions $\partial_{[\mu} f_{\nu]} = \partial_\mu f^\mu = 0$. The constant solution cannot lead to a normalizable zero mode[7] and must therefore be discarded. The possibly normalizable solutions to $\not{D}_k \bar\lambda = 0$ are then given by

$$\bar\lambda = 2i\, e^{k\phi}\, \partial_\mu h\, \bar\sigma^\mu \xi \ , \tag{7.5}$$

with ξ a constant spinor. For spherical symmetry, the only harmonic function available is $e^{-\phi}$. Looking at the asymptotic behavior, one finds that the only normalizable zero modes are

$$\bar\lambda = 2i\, e^{k\phi}\, \partial_\mu e^{-\phi}\, \bar\sigma^\mu \xi \ , \qquad k \geq \tfrac{3}{4} \ . \tag{7.6}$$

Comparing with the hyperino zero mode equations (7.1) and (7.2), we conclude that

$$\bar\lambda^2 = 0 \ , \qquad \lambda^1 = 0 \ , \tag{7.7}$$

since $k = -3/4$. Zero modes for $\bar\lambda^1$ and λ^2 would look like (7.6) with the smallest possible value for k if no gravitinos were present. Before proceeding, we will determine the gravitinos from their own field equations. The solution we then plug into (7.1), (7.2) and solve for $\bar\lambda^1$ and λ^2.

No spin 3/2 zero modes

In the instanton background, (5.9) becomes

$$i\varepsilon^{\mu\nu\rho\sigma}\sigma_\rho \begin{pmatrix} \partial_\sigma \bar\psi_{\nu 1} + \tfrac{1}{4}\partial_\sigma\phi\,\bar\psi_{\nu 1} - e^{-\phi/2}\partial_\sigma\chi\,\bar\psi_{\nu 2} \\ \partial_\sigma\bar\psi_{\nu 2} - \tfrac{1}{4}\partial_\sigma\phi\,\bar\psi_{\nu 2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{-\phi/2}\partial_\nu\chi \\ \partial_\nu\phi \end{pmatrix}\sigma^\nu\bar\sigma^\mu\lambda^1 \ . \tag{7.8}$$

Notice that only λ^1 couples to the gravitino; this is consistent with the hyperino equations of motion as one can easily check. But we have just concluded in (7.7) that λ^1 vanishes in the instanton background, i.e. there is no zero mode in λ^1. This means that the inhomogeneous term on the right-hand side of the gravitino equations vanishes, and that the second gravitino decouples from the first. Writing

$$\bar\psi_{\mu 2} = e^{\phi/4}\bar\zeta_{\mu 2} \ , \tag{7.9}$$

we can, using some sigma matrix identities such as (A.4), rewrite the second gravitino equation of motion as

$$\sigma^\mu(\partial_\mu\bar\zeta_{\nu 2} - \partial_\nu\bar\zeta_{\mu 2}) = 0 \ . \tag{7.10}$$

Of course, there are an infinite number of solutions to this equation, for example $\bar\zeta_\mu = \partial_\mu\bar\zeta$, but we still have to subject them to the supersymmetry gauge-fixing procedure. We adopt the common gauge

$$\bar\sigma^\mu\psi_\mu^i = \sigma^\mu\bar\psi_{\mu i} = 0 \ , \tag{7.11}$$

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[7]Normalizable zero modes Z must satisfy $\int_0^\infty dr\, r^3 |Z|^2 < \infty$. This implies that the asymptotic behavior of Z at infinity must go to zero like $r^{-5/2}$ or faster, and at the origin Z may not diverge faster than $r^{-3/2}$.

in particular $\sigma^\mu \bar\zeta_{\mu 2} = 0$. Then, (7.10) further simplifies to $\bar\partial\!\!\!/\, \bar\zeta_{\mu 2} = 0$, still subject to (7.11). The solution is given by

$$\bar\zeta_{\mu 2} = \partial_\mu \partial_\nu h\, \bar\sigma^\nu \xi \ , \tag{7.12}$$

with ξ a constant spinor and h a harmonic function. For a given solution, one can still act with residual supersymmetry transformations that preserve the gauge (7.11). The gravitino supersymmetry transformations in the NS5-brane instanton (upper sign in (3.1)) background read

$$\delta_\epsilon \bar\psi_{\mu 1} = 2(\partial_\mu + \tfrac{1}{4}\partial_\mu \phi)\bar\epsilon_1 - 2\mathrm{e}^{-\phi/2}\,\partial_\mu \chi\, \bar\epsilon_2$$
$$\delta_\epsilon \bar\psi_{\mu 2} = 2(\partial_\mu - \tfrac{1}{4}\partial_\mu \phi)\bar\epsilon_2 \ . \tag{7.13}$$

Such a supersymmetry transformation has the effect of shifting $\bar\zeta_{\mu 2}$ with a total derivative, so it obviously still satisfies (7.10). Imposing furthermore the gauge-fixing condition implies that the residual supersymmetry transformations must satisfy

$$\bar\partial\!\!\!/\,\big(\mathrm{e}^{-\phi/4}\bar\epsilon_2\big) = 0 \ . \tag{7.14}$$

The solutions are again given by a constant or by the derivative of a harmonic function and acting with such a residual supersymmetry transformation remains inside the class given by (7.12). Therefore the only candidate gravitino zero mode for $\bar\psi_{\mu 2}$ is given by

$$\bar\psi_{\mu 2} = \mathrm{e}^{\phi/4}\partial_\mu \partial_\nu h\, \bar\sigma^\nu \xi \ , \tag{7.15}$$

where h can again be chosen to be $h = \mathrm{e}^{-\phi}$ for spherically symmetric harmonic functions. However, this zero mode is not normalizable, since at the origin it diverges too fast, as one can explicitly check. As a consequence, there is no normalizable gravitino zero mode for $\bar\psi_{\mu 2}$, and we can set it to zero in the equation of motion for the first gravitino. A similar analysis now shows that there is no normalizable zero mode for $\bar\psi_{\mu 1}$ either, and so we can set both gravitinos to zero in the hyperino equations for λ^2 and $\bar\lambda^1$. The hyperino zero modes then follow from the discussion above.

Index theorems?

We conclude that the solutions of the linearized fermion equations of motion in the presence of the fivebrane instanton are given by

$$\bar\psi_{\mu i} = \psi^i_\mu = \lambda^1 = \bar\lambda^2 = 0 \ , \tag{7.16}$$

and the four zero modes lie in

$$\bar\lambda^1 = 2\mathrm{i}\,\mathrm{e}^{3\phi/4}\,\partial_\mu \mathrm{e}^{-\phi}\,\bar\sigma^\mu \xi \ , \qquad \lambda^2 = 2\mathrm{i}\,\mathrm{e}^{3\phi/4}\,\partial_\mu \mathrm{e}^{-\phi}\,\sigma^\mu \bar\xi \ , \tag{7.17}$$

where the four fermionic collective coordinates are denoted by the two unrelated spinors ξ and $\bar{\xi}$. Notice the difference with Yang-Mills theories, where fermionic zero modes only appear in one chiral sector.

These four fermionic zero modes should be counted by the index of the operators

$$\slashed{D}_k \equiv e^{k\phi}\slashed{\partial}\,e^{-k\phi}\,, \qquad \bar{\slashed{D}}_k \equiv e^{k\phi}\bar{\slashed{\partial}}\,e^{-k\phi}\,, \qquad k \in \mathbb{R}\,, \tag{7.18}$$

where $\slashed{\partial} = \sigma^\mu \partial_\mu$ and $\bar{\slashed{\partial}} = \bar{\sigma}^\mu \partial_\mu = \slashed{\partial}^\dagger$, see (7.3). Clearly, these operators are not hermitean, but satisfy $(\slashed{D}_k)^\dagger = \bar{\slashed{D}}_{-k}$. Due to this property, they are different from the usual Dirac operators with anti-hermitean connection. The case we are interested in corresponds to $k = 3/4$. The relevant indices are defined as

$$\mathrm{Ind}\,\slashed{D}_k \equiv \dim\ker\slashed{D}_k - \dim\ker\bar{\slashed{D}}_{-k}\,, \qquad \mathrm{Ind}\,\bar{\slashed{D}}_k \equiv \dim\ker\bar{\slashed{D}}_k - \dim\ker\slashed{D}_{-k}\,. \tag{7.19}$$

In the beginning of this section we have shown explicitly that both $\slashed{D}_{-3/4}$ and $\bar{\slashed{D}}_{-3/4}$ have no normalizable zero modes, so the index for $k = 3/4$ indeed counts the number of zero modes. Index theorems for our kind of operators and spacetime topology should therefore reproduce that $\mathrm{Ind}\,\bar{\slashed{D}}_{3/4} = \mathrm{Ind}\,\slashed{D}_{3/4} = 2$. It would be interesting to find and apply such index theorems to instanton solutions in scalar-tensor theories in general.

Broken supersymmetries

We now show that these zero modes are precisely generated by acting with the residual supersymmetries (7.14) on the hyperinos. As already said, the solutions for $e^{-\phi/4}\bar{\epsilon}_2$ are either constants or given in terms of a harmonic function. The latter lead to non-normalizable gravitino zero modes which we have discarded, but the constant spinor solution leaves the second gravitino invariant,

$$\bar{\epsilon}_2 = e^{\phi/4}\,\bar{\eta}\,, \tag{7.20}$$

with $\partial_\mu\bar{\eta} = 0$. The broken supersymmetries for $\bar{\epsilon}_1$ are then determined by those transformations that leave $\bar{\psi}_{\mu 1}$ invariant and preserve the gauge condition (7.11), but which act nontrivially on the spin $1/2$ fermions. Using (7.20) we find

$$\bar{\epsilon}_1 = e^{-\phi/4}\big(\chi\bar{\eta} + \bar{\xi}'\big)\,, \tag{7.21}$$

where $\bar{\xi}'$ is another constant spinor. With the $\bar{\epsilon}_i$ inserted into the λ^a transformations, we find that, after a redefinition $\bar{\xi} = \bar{\xi}' + \chi_0\bar{\eta}$, the Killing spinor proportional to $\bar{\eta}$ leaves the fields invariant and thus generates unbroken supersymmetries, as observed in section 6. The Killing spinor proportional to $\bar{\xi}$ on the other hand yields the λ-zero mode[8]

$$^{(1)}\lambda^1 = 0\,, \qquad ^{(1)}\lambda^2 = -2\mathrm{i}\,e^{-\phi/4}\,\partial_\mu\phi\,\sigma^\mu\bar{\xi}\,. \tag{7.22}$$

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[8]The left superscript counts the Grassmann collective coordinates (GCC) in the fields (excluding the supersymmetry parameters). We omit the superscript $^{(0)}$.

Using the relation between Killing spinors of opposite chirality derived in the previous section, we immediately obtain the unbarred broken supersymmetries:

$$\epsilon^1 = 0 , \qquad \epsilon^2 = -e^{-\phi/4}\xi . \tag{7.23}$$

Here ξ is another constant spinor. The zero mode generated by ϵ^2 is found to be

$$^{(1)}\bar{\lambda}^1 = -2i\, e^{-\phi/4} \, \partial_\mu \phi \, \bar{\sigma}^\mu \xi , \qquad ^{(1)}\bar{\lambda}^2 = 0 . \tag{7.24}$$

Notice that this is a function of space x and of the position x_0 of the instanton. One can diagrammatically represent this fermionic zero mode by a line connecting the fermion at x to the instanton at the point x_0, cf. (9.7).

Similar results can be obtained for the anti-instanton, the zero modes will now be in λ^1 and $\bar{\lambda}^2$. For further convenience below, we introduce a handy notation that connects the hyperino-labels "1" and "2" with the (anti-) instanton-labels "+" and "−". This is done in such a way that the hyperino labels 1 and 2 are denoted by upper and lower indices respectively. These indices can then be further specified by indicating the background, instanton or anti-instanton. In this notation, the absence of fermionic zero modes is expressed by the equations $^{(1)}\lambda^\pm = 0$, where the upper index is associated to the first hyperino in the instanton $(+)$ background, and the lower label is associated to the second hyperino in the anti-instanton $(-)$ background. Similarly we have that $^{(1)}\bar{\lambda}^\mp = 0$. For the broken supersymmetries we have $\epsilon^\pm = \bar{\epsilon}_\mp = 0$. In the zero mode sector we can write

$$^{(1)}\lambda^\mp = -2i\, e^{-\phi/4} \, \partial_\mu \phi \, \sigma^\mu \bar{\xi}^\mp , \qquad ^{(1)}\bar{\lambda}^\pm = \mp 2i\, e^{-\phi/4} \, \partial_\mu \phi \, \bar{\sigma}^\mu \xi^\pm . \tag{7.25}$$

The fermionic collective coordinates $\bar{\xi}^\mp$ are two independent two-component spinors which distinguish between instantons and anti-instantons. Similarly for $\xi^\pm$, and we have have put in an additional sign for the anti-instanton fermionic zero mode in $\bar{\lambda}^2$ for later convenience. For the broken supersymmetries, we have

$$\epsilon^\mp = -e^{-\phi/4}\, \xi^\mp , \qquad \bar{\epsilon}_\pm = \pm e^{-\phi/4}\, \bar{\xi}^\pm . \tag{7.26}$$

In further sections, we sometimes drop these $\pm$ indices on ξ and $\bar{\xi}$, to avoid heavy notation and because it is always clear from the context what is meant.

8 Measure on the instanton moduli space

From field theory instanton calculations, we learn that the path integral measure reduces to a finite dimensional integral over the collective coordinates (the instanton moduli space), together with a path integral over the quantum fluctuations around the instanton. The measure on the instanton moduli space of collective coordinates is obtained from computing

the inner products of the zero modes [34], see also [35] for a review. Semiclassically, the integration over the quantum fluctuations yields one-loop determinants that have to be evaluated in the instanton background. Finding this measure is an essential step for computing correlation functions nonperturbatively.

How much of these field theory techniques can be applied to NS5-brane instantons? First, we should keep in mind that we are approximating the wrapped fivebrane by a four-dimensional supergravity instanton solution. The (bosonic) collective coordinates are then just the positions of the instanton in four-dimensional Euclidean space. In string theory, the NS5-brane instanton is described by an embedding of the worldvolume into the ten-dimensional product space $\mathbb{R}^4 \times \mathrm{CY}$, such that the worldvolume of the fivebrane is wrapped around the entire CY. The maps describing the embedding are thought of as the collective coordinates of the instanton. Integrating over the moduli space would then involve doing a path-integral over the worldvolume theory in the supergravity background $\mathbb{R}^4 \times \mathrm{CY}$. This is the general strategy advocated in [7]. This procedure is cumbersome, however, due to the complicated nature of the NS-fivebrane worldvolume theory. In this paper, we have not included any such worldvolume effects; we describe here the moduli space as the finite dimensional space with coordinates x_0^μ and their fermionic partners ξ and $\bar{\xi}$ as discussed in the previous section.

Second, there are the one-loop determinants. In our approach, these determinants should be computed in the supergravity theory. They receive corrections not only from the hypermultiplet fluctuations, but also from the fluctuations of the gravitational and $h_{1,1}$ vector multiplets. Their classical values in the instanton background are trivial (flat metric, and vanishing vector multiplets), but their quantum effects cannot be ignored. This is a complicated calculation which lies beyond the scope of this paper. Moreover, these (and higher-) loop effects would have to be computed in the full ten-dimensional string theory; it remains to be seen if such a calculation can be done. As we indicate below, we shall simply denote the determinants by K and leave them unspecified in further calculations. Clearly a better understanding of instanton calculations in string theory is needed. We have taken here a more pragmatic approach, and follow a line of thinking somewhat similar to what Salviati and Sagredo conclude from their discussion in [36].

We now proceed to calculate the measure. The zero modes are, by definition, the zero eigenvalue eigenfunctions of the operator sandwiched between the one-loop quantum fluctuations, and can be obtained by taking the derivative of the instanton solution with respect to the collective coordinates. We will here carry out the procedure of computing the metric on the moduli space of single-centered instantons. The case of anti-instantons is completely analogous.

There are two modifications with respect to the calculation in [34]. First, we are dealing

with a non-linear sigma model in the scalar field sector, where nontrivial metrics $\mathcal{G}_{AB}$ and M^{IJ} appear. Second, we are working with 2-form tensor fields which have to be properly gauge-fixed, and whose zero modes will need to satisfy a corresponding background gauge condition. In the first part of this section we repeat the analysis of [34], applied to our system, and in the second part we deal with the fermionic sector.

Bosonic measure

If we denote the bosonic fields of the double-tensor multiplet collectively by

$$\Phi^M = \{\phi^A, B_{\mu\nu I}\} , \tag{8.1}$$

and expand the action about the instanton solution to quadratic order in the fluctuations

$$\Phi^M = \Phi^M_{\text{cl}} + \Phi^M_{\text{qu}} , \tag{8.2}$$

we can write (with flat background metric $g_{\mu\nu} = \delta_{\mu\nu}$)

$$S = S_{\text{cl}} + \frac{1}{2} \int d^4x \, \Phi^M_{\text{qu}} \, \mathcal{G}_{MP} \Delta^P{}_N \, \Phi^N_{\text{qu}} + O(\Phi^3_{\text{qu}}) . \tag{8.3}$$

Here, we have denoted (suppressing spacetime indices in the tensor sector)

$$\mathcal{G}_{MN} = \begin{pmatrix} \mathcal{G}_{AB}(\phi_{\text{cl}}) & 0 \\ 0 & M^{IJ}(\phi_{\text{cl}}) \end{pmatrix} . \tag{8.4}$$

$\Delta^M{}_N$ is a hermitean operator with respect to the inner product defined by $\mathcal{G}_{MN}$, and can be determined by explicitly expanding in the fluctuations. For the moment it suffices to say that it takes the form (again suppressing spacetime indices)

$$\Delta^M{}_N = \begin{pmatrix} -\delta^A{}_B \, \partial^2 + \dots & * \\ * & -\delta_I{}^J \, \partial^2 + \dots \end{pmatrix} , \tag{8.5}$$

where the ellipsis stands for terms with operators at most linear in derivatives. The off-diagonal terms are not written explicitly, but can also be seen to be at most linear in derivatives.

Clearly, this operator is some sort of generalized Laplace operator, and we assume it has a basis of orthogonal eigenfunctions F_i^M in which we expand the fluctuations,

$$\Delta^M{}_N F_i^N = \varepsilon_i F_i^M , \qquad \Phi^M_{\text{qu}} = \sum_i \xi_i F_i^M . \tag{8.6}$$

The zero modes $F_{i_0}^M$ are eigenfunctions with zero eigenvalues $\varepsilon_{i_0} = 0$, but nonzero fluctuation coefficients ξ_{i_0}. The norms of the eigenfunctions are taken with respect to the metric $\mathcal{G}_{MN}$,

$$U_{ij} \equiv \langle F_i \, | \, F_j \rangle \equiv \int d^4x \, F_i^M \mathcal{G}_{MN} F_j^N . \tag{8.7}$$

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The metric on the moduli space is found by computing the inner product of the zero modes, and the latter can be obtained by taking derivatives of the instanton solution with respect to the collective coordinates[9]. In the case of the single-centered instanton, these are just the positions x_0^μ of the instanton in $\mathbb{R}^4$. Thus, there are four zero modes and the moduli space metric $(U_0)_{\mu\nu}$ is four-dimensional, with contributions from both the scalars and tensors. Since $\mathcal{G}_{MN}$ is block-diagonal, there is no mixing between the zero modes of these two sectors.

In the scalar sector the metric $\mathcal{G}_{AB}$ is diagonal, so we can consider the contributions from ϕ and χ separately. For the dilaton zero mode $\partial\phi/\partial x_0^\mu = e^\phi \partial_\mu e^{-\phi}$ we find, using the spherical symmetry of the single-centered instanton,

$$U_{\mu\nu}^{(\phi)} = \int \mathrm{d}^4 x \, \frac{\partial \phi}{\partial x_0^\mu} \frac{\partial \phi}{\partial x_0^\nu} = \int \mathrm{d}^4 x \, \frac{x_\mu x_\nu}{r^2} \, e^{2\phi}(\partial_r e^{-\phi})^2 = \frac{1}{4} \delta_{\mu\nu} \int \mathrm{d}^4 x \, e^{2\phi}(\partial_r e^{-\phi})^2 \; . \qquad (8.8)$$

The integral formula (E.1) now yields the result

$$U_{\mu\nu}^{(\phi)} = \frac{|Q_2|}{4\,g_s^2} \, \delta_{\mu\nu} \; . \qquad (8.9)$$

Analogously, from (6.6) it follows for the χ zero mode $\partial\chi/\partial x_0^\mu = g_s^2 \Delta\chi \, e^{2\phi} \partial_\mu e^{-\phi}$ that

$$U_{\mu\nu}^{(\chi)} = \int \mathrm{d}^4 x \, e^{-\phi} \frac{\partial \chi}{\partial x_0^\mu} \frac{\partial \chi}{\partial x_0^\nu} = \frac{|Q_2|}{8} \, (\Delta\chi)^2 \, \delta_{\mu\nu} \; . \qquad (8.10)$$

Here the $e^{-\phi}$ insertion is the $\mathcal{G}_{\chi\chi}$-component of the metric (8.4).

The complete moduli space metric is the sum of the above Us and those of the tensors, which we shall compute now. Being subject to gauge symmetries, we first have to gauge-fix the tensors. We shall impose the background gauge condition

$$\partial^\mu \big(M^{IJ} B_{\mu\nu J}^{\mathrm{qu}} \big) = 0 \; . \qquad (8.11)$$

This requires a corresponding modification of $\Delta^M{}_N$ and the inclusion of ghosts. The instanton configurations are solutions to the classical, gauge-invariant equations of motion only, so derivatives with respect to the collective coordinates in general do not yield zero modes of the gauge-fixed operator $\Delta^M{}_N$. We need to add a suitable gauge transformation to keep them in the background gauge,

$$Z_{\mu\nu I\rho} = \frac{\partial B_{\mu\nu I}}{\partial x_0^\rho} - 2 \, \partial_{[\mu} \Lambda_{\nu] I\rho} \; . \qquad (8.12)$$

With $\Lambda_{\nu I\rho} = B_{\nu\rho I}$, we obtain

$$Z_{\mu\nu I\rho} = -H_{\mu\nu\rho I} = \varepsilon_{\mu\nu\rho\sigma} H_I^\sigma \; . \qquad (8.13)$$

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[9] While such a derivative always gives a zero mode (modulo gauge transformations, see below), it is to our knowledge unclear whether in general there could be more zero modes than collective coordinates.

These $Z_{\mu\nu I\rho}$ satisfy the gauge condition (8.11) by virtue of the classical tensor field equations. It follows that they are zero modes of the gauge-fixed $\Delta^M{}_N$. Note that we do not have to solve explicitly for the gauge potentials to compute their zero mode norms, knowing the field strengths is sufficient. We now calculate

$$U_{\mu\nu}^{(B)} = \frac{1}{2} \int \mathrm{d}^4 x \, M^{IJ} Z^{\rho\sigma}{}_{I\mu} Z_{\rho\sigma J\nu} = \int \mathrm{d}^4 x \, M^{IJ} (\delta_{\mu\nu} H_I^\rho H_{\rho J} - H_{\mu I} H_{\nu J}) \ . \tag{8.14}$$

Using spherical symmetry and the Bogomol'nyi condition (3.1), we obtain

$$U_{\mu\nu}^{(B)} = \frac{3}{4} \delta_{\mu\nu} \int \mathrm{d}^4 x \, (\mathrm{e}^\phi \hat{H}_1^\rho \hat{H}_{\rho 1} + \mathrm{e}^{2\phi} H_2^\rho H_{\rho 2}) = 3 \big(U_{\mu\nu}^{(\chi)} + U_{\mu\nu}^{(\phi)} \big) \ . \tag{8.15}$$

The sum of the scalar and tensor parts finally gives

$$(U_0)_{\mu\nu} = U_{\mu\nu}^{(\phi)} + U_{\mu\nu}^{(\chi)} + U_{\mu\nu}^{(B)} = S_{\mathrm{cl}} \, \delta_{\mu\nu} \ , \tag{8.16}$$

with S_{cl} as in (3.10), a result also familiar from Yang-Mills instantons!

The bosonic part of the single-centered (anti-) instanton moduli space measure is therefore

$$\int \frac{\mathrm{d}^4 x_0}{(2\pi)^2} \, (\det U_0)^{1/2} \, \mathrm{e}^{-S_{\mathrm{inst}}^\pm} (\det{}' \Delta)^{-1/2} = \int \frac{\mathrm{d}^4 x_0}{(2\pi)^2} \, S_{\mathrm{cl}}^2 \, \mathrm{e}^{-S_{\mathrm{inst}}^\pm} (\det{}' \Delta)^{-1/2} \ , \tag{8.17}$$

where $\det{}' \Delta$ stands for the amputated determinant, the product of all nonzero eigenvalues. As explained in the beginning of this section, computing this determinant lies beyond the scope of this paper.

Fermionic measure

We now expand the fermion terms in the action about the (bosonic) instanton solution, up to quadratic order in the fluctuations. Similarly to the bosonic case, it is sufficient to consider first fluctuations in the matter fermions λ^a only, and freeze the fluctuations from other multiplets, such as those coming from the gravitinos ψ_μ^i. We can then write

$$S_2 = \int \mathrm{d}^4 x \, \mathrm{i} \, \lambda_{\mathrm{qu}}^a (\slashed{D}_{3/4})_{a\bar{b}} \, \bar{\lambda}_{\mathrm{qu}}^{\bar{b}} \tag{8.18}$$

for the quadratic part of the action, where

$$(\slashed{D}_{3/4})_{a\bar{b}} = \begin{pmatrix} \slashed{D}_{3/4} & 0 \\ 0 & \slashed{D}_{-3/4} \end{pmatrix} \ , \tag{8.19}$$

and $\slashed{D}_k$ is defined in (7.3). We have shown in sections 5 and 7 that (8.18) indeed produces the correct field equations.

We can construct hermitean operators $M_k = \slashed{D}_{-k} \bar{\slashed{D}}_k$ and $\bar{M}_k = \bar{\slashed{D}}_{-k} \slashed{D}_k$. The cases of interest are when $k = \pm 3/4$. The spectrum of nonzero modes of $M_{-3/4}$ and $\bar{M}_{3/4}$ is

identical, and similarly for $M_{3/4}$ and $\bar{M}_{-3/4}$. This can be seen as follows: Let F_i^1 and F_i^2 denote a basis of eigenfunctions of $M_{-3/4}$ and $M_{3/4}$ respectively, and $\bar{F}_i^1$ and $\bar{F}_i^2$ a basis of eigenfunctions of $\bar{M}_{3/4}$ and $\bar{M}_{-3/4}$ respectively. The eigenfunctions of $M_{-3/4}$ and $\bar{M}_{3/4}$ are then related, with the same eigenvalue $\varepsilon_i^1 = \bar{\varepsilon}_i^1 \neq 0$, by $\bar{F}_i^1 = (\varepsilon_i^1)^{-1/2} \bar{\slashed{D}}_{-3/4} F_i^1$, or inversely, $F_i^1 = (\varepsilon_i^1)^{-1/2} \slashed{D}_{3/4} \bar{F}_i^1$. Similarly, the spectrum of nonzero modes of $M_{3/4}$ and $\bar{M}_{-3/4}$ is identical, and the relation between the eigenfunctions is given by $\bar{F}_i^2 = (\varepsilon_i^2)^{-1/2} \slashed{D}_{3/4} F_i^2$, with inverse $F_i^2 = (\varepsilon_i^2)^{-1/2} \slashed{D}_{-3/4} \bar{F}_i^2$. We here assumed for simplicity that the eigenvalues are positive, the argument is similar for negative eigenvalues. Bearing in mind that both $M_{3/4}$ and $\bar{M}_{3/4}$ have zero modes, together with the fact that the fermion zero modes are in λ^2 and $\bar{\lambda}^1$, we can expand the fermions in a basis of eigenfunctions (suppressing spinor indices),

$$\lambda_{\text{qu}}^a = \sum_i \xi_i^a F_i^a , \qquad \bar{\lambda}_{\text{qu}}^{\bar{a}} = \sum_i \bar{\xi}_i^{\bar{a}} \bar{F}_i^{\bar{a}} , \tag{8.20}$$

with ξ_i^a and $\bar{\xi}_i^a$ anticommuting (there is no sum over a). Plugging this into the action, and using the relation between the different eigenfunctions as discussed above, we get

$$S_2 = \mathrm{i} \sum_{a,i,j} \xi_i^a U_{ij}^{aa} (\varepsilon_j^a)^{1/2} \bar{\xi}_j^a , \qquad U_{ij}^{ab} \equiv \int \mathrm{d}^4 x \ F_i^a F_j^b . \tag{8.21}$$

We then define the fermionic part of the path-integral measure as (up to a sign from the ordering of the differentials)

$$[\mathrm{d}\lambda] \, [\mathrm{d}\bar{\lambda}] \equiv \prod_a \prod_i \mathrm{d}\xi_i^a \, \mathrm{d}\bar{\xi}_i^a \, (\det U^{aa})^{-1} , \tag{8.22}$$

such that the fermion integral gives the Pfaffians of $\bar{\slashed{D}}_{3/4}$ and $\slashed{D}_{3/4}$ in the nonzero mode sector. In the zero mode sector, we are left over with an integral over the four GCC, which are combined into two spinors, multiplied by the inverses of the norms of the zero modes. These zero mode eigenfunctions have the form $Z_{\alpha\beta'}^2 = \partial^{(1)}\lambda_\alpha^2 / \partial\bar{\xi}^{\beta'}$ given in (7.22), so that we find for their inner product

$$U_{\alpha'\beta'}^{22} = \int \mathrm{d}^4 x \, Z^{2\gamma}{}_{\alpha'} \, Z^2_{\gamma\beta'} = -4 \int \mathrm{d}^4 x \, \mathrm{e}^{-\phi/2} \partial_\mu \phi \, \partial_\nu \phi \, (\varepsilon \, \bar{\sigma}^\mu \sigma^\nu)_{\alpha'\beta'}$$
$$= 4 \, \varepsilon_{\alpha'\beta'} \int \mathrm{d}^4 x \, \mathrm{e}^{3\phi/2} (\partial_r \mathrm{e}^{-\phi})^2 = \frac{8 \, |Q_2|}{g_s} \, \varepsilon_{\alpha'\beta'} . \tag{8.23}$$

The fermionic measure on the moduli space of collective coordinates then is

$$\int \mathrm{d}^2 \xi \, \mathrm{d}^2 \bar{\xi} \left(\frac{g_s}{8 \, |Q_2|} \right)^2 \left(\det{}' M_{3/4} \, \det{}' \bar{M}_{3/4} \right)^{1/2} . \tag{8.24}$$

Here, our convention is that $\mathrm{d}^2 \xi \equiv \mathrm{d}\xi_1 \, \mathrm{d}\xi_2$.

The complete measure is then given by (8.24) combined with (8.17), and has still to be supplemented with the gravitino, supersymmetry ghost, and other fermionic determinants

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from the vector multiplets. In the following we denote with $K_{\text{1-loop}}^{\pm}$ the ratio of all fermionic and bosonic determinants in the one-(anti-)instanton background. The single-centered (anti-) instanton measure is then

$$\int \frac{\mathrm{d}^4 x_0}{(2\pi)^2} \int \mathrm{d}^2\xi \, \mathrm{d}^2\bar{\xi} \left(\frac{g_s \, S_{\text{cl}}}{8 \, |Q_2|}\right)^2 K_{\text{1-loop}}^{\pm} \, \mathrm{e}^{-S_{\text{inst}}^{\pm}} \ . \tag{8.25}$$

This measure is the starting point for computing instanton corrections to certain correlators. All the preparation is now done, and we can finally focus on the explicit calculation of correlation functions.

9 Correlation functions

We have found in previous sections that there are four Grassmann collective coordinates (GCC), and that they are all associated to the broken supersymmetries. Also, the path integral measure contains an integration over all collective coordinates, including the GCC. Hence, a generic correlation function will only be non-zero if the operators inserted in the path integral saturate the GCCs of the measure. It is then clear that there will be a non-zero four-point fermion correlation function. Diagrammatically, such a four-point vertex consists of four fermion zero modes connected to an instanton at position x_0 which is integrated over. Computing this diagram, one could read off the 4-index tensor that determines the four-fermi terms in the effective action. As explained in section 4, this procedure is a bit complicated due to the fact that we are working in a 1.5 order formalism, where additional four-fermi terms are hidden in the spacetime curvature scalar $R(\omega)$ as a function of the spin connection. Moreover, the four-fermi correlator would merely determine (target space) curvature-like terms, rather than the objects M^{IJ}, $\mathcal{G}_{AB}$ and A_A^I which we are really interested in. Luckily, there is a way out of this by studying the GCC dependence of the scalars and tensors.

GCC dependence of scalars and tensors

We can in fact compute the instanton corrections to M^{IJ}, $\mathcal{G}_{AB}$ and A_A^I more directly by using again the broken supersymmetries. These generate fluctuations which, by supersymmetry, are related to the purely bosonic instanton, and are genuine zero modes which leave the instanton action unchanged. The infinitesimal broken supersymmetries at linear order induce the fermionic zero modes discussed above. The full broken supersymmetry group is found by exponentiating the infinitesimal transformations (4.4), and acting with them on the bosonic instanton, we induce a GCC dependence of the scalars and tensors. Expanding to second order, there will be a quadratic GCC dependence, and this is sufficient for our purpose. The relevant correlators then will be 2- and 3-point functions of scalars and

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tensors. Diagrammatically, they correspond to instanton corrected propagators. These diagrams are related by supersymmetry to the above mentioned four-fermi vertex.

At second order in the GCC, the scalars are given by

$$^{(2)}\phi^A = \frac{1}{2}\,\delta_\epsilon^2 \phi^A|_{\rm cl} = \frac{1}{\sqrt{2}}\big(\gamma_{ia}^A(\phi_{\rm cl})\,\epsilon^{i\,(1)}\lambda^a + \bar{\gamma}_{\bar{a}}^{iA}(\phi_{\rm cl})\,\bar{\epsilon}_i\,^{(1)}\bar{\lambda}^{\bar{a}}\big)\;. \tag{9.1}$$

The numerical factors come from the fact that the second supersymmetry variation comes with a factor of $1/2$, and from the rescaling of ϵ by a factor of $\sqrt{2}$, cf. footnote 4. Using the notation introduced at the end of section 7, we find that, since $^{(1)}\lambda^\pm = {}^{(1)}\bar{\lambda}^\mp = 0$ and for the broken supersymmetries $\epsilon^\pm = \bar{\epsilon}_\mp = 0$, only terms proportional to $\gamma_{\mp\mp}^A$ and $\bar{\gamma}_{\pm}^{\pm A}$ contribute. For the dilaton these are both zero, so only χ gets corrected at this order:

$$^{(2)}\chi = 2\mathrm{i}\,\partial_\mu\phi\,\xi\sigma^\mu\bar{\xi}\;, \qquad ^{(2)}\phi = 0\;. \tag{9.2}$$

Due to our conventions for the fermionic zero modes chosen in (7.25), this expression for χ is the same in the instanton and anti-instanton background.

Analogously, the second order corrections of the tensors follow from (4.7). The instanton and anti-instanton cases yield, up to a sign, the same answer,

$$^{(2)}B_{\mu\nu 1} = \mp 2\mathrm{i}\,\varepsilon_{\mu\nu\rho\sigma}\,\partial^\rho \mathrm{e}^{-\phi}\,\xi\sigma^\sigma\bar{\xi}\;, \qquad ^{(2)}B_{\mu\nu 2} = 0\;. \tag{9.3}$$

Notice again that only the RR sector is turned on.

It turns out that the Bogomol'nyi equation (3.1) still holds at this order in the GCC, for one can easily check that

$$^{(2)}H_{\mu 1} = \pm\partial_\mu\big(\mathrm{e}^{-\phi\,(2)}\chi\big)\;. \tag{9.4}$$

The second component of (3.1) is trivially satisfied. It might surprise the reader, who is somewhat familiar with instanton calculus, that the equations of motion are satisfied without any fermion-bilinear source term. One would expect such a source term to be present, since (9.2) and (9.3) are obtained by acting with the broken supersymmetries that also generate the fermionic zero modes. This is typically what happens with the Yukawa terms in $N = 2$ or $N = 4$ SYM theory in flat space; there the adjoint scalar field is found from solving the inhomogeneous Laplace equation with a fermion-bilinear source term. The fermionic zero modes in the presence of a YM instanton then determine the profile and GCC dependence of the adjoint scalar field. Some references where this is discussed in more detail are given in [37].

In the case at hand, the fermion bilinear source term actually vanishes when the zero modes are plugged in. To see this, let us first consider the tensors, for which the full equations of motion read

$$e^{-1}\frac{\delta S}{\delta B_{\mu\nu I}} = \varepsilon^{\mu\nu\rho\sigma}\partial_\rho\Big[M^{IJ}\mathcal{H}_{\sigma J} - \mathrm{i}A_A^I\partial_\sigma\phi^A + \frac{\mathrm{i}}{\sqrt{2}}M^{IJ}(g_{Jia}\psi_\sigma^i\lambda^a + \text{c.c.})$$

$$+ \mathrm{i}M^{IJ}k_{Ja\bar{a}}\,\lambda^a\sigma_\sigma\bar{\lambda}^{\bar{a}}\Big]\;. \tag{9.5}$$

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The fermionic zero modes we have found above do not enter these equations directly, because $^{(1)}\psi^i_\mu = {}^{(1)}\bar\psi_{\mu i} = 0$, and the two matrices $M^{IJ} k_{Ja\bar a}$ are diagonal (actually zero for $I = 1$), but for $a = \bar a$ either $^{(1)}\lambda^a$ or $^{(1)}\bar\lambda^{\bar a}$ vanishes. Hence, up to second order in the GCC, only the bosonic fields contribute. This is consistent with the fact that the BPS condition still holds at this order. A similar analysis can be done for the equations of motion for the scalars.

2-point functions

The relevant objects for computing correlation functions are products of operators which saturate the GCC integral. These can be bilinears in the RR fields $^{(2)}\chi$ and $^{(2)}H^\mu_1$, a combination of one RR field with the zero modes $^{(1)}\lambda^\mp$ and $^{(1)}\bar\lambda^\pm$, or the product of all four zero modes. Moreover, an arbitrary number of background fields may be inserted. As explained in the beginning of this section, we shall not compute the fermionic 4-point function, since we can learn much more from the 2- and 3-point functions.

The first step is to take the large distance limit of the zero modes and express them in terms of propagators, which will enable us to read off the effective vertices from the correlation functions by stripping off the external legs. For the bosons we find

$$^{(2)}\chi(x) = -2\mathrm{i}\,|Q_2|\,g_s^{-2}\,\xi\sigma^\mu\bar\xi\,\partial_\mu G(x, x_0)\left(1 + \dots\right)$$
$$^{(2)}H^\mu_1(x) = \mp 2\mathrm{i}\,|Q_2|\,\xi\sigma^\nu\bar\xi\left(\partial^\mu\partial_\nu - \delta^\mu_\nu\partial^2\right)G(x, x_0)\,, \tag{9.6}$$

where $G(x, x_0) = 1/4\pi^2(x - x_0)^2$ is the massless scalar propagator. The second equation is exact, while from the first we only keep the leading term in the large distance expansion valid when $(x - x_0)^2 \gg |Q_2|/4\pi^2 g_s^2$. In this limit the dilaton is effectively given by $\mathrm{e}^{-\phi} \approx \mathrm{e}^{-\phi_\infty} = g_s^2$, and similarly $\chi \approx \chi_\infty$, so the fields are replaced by their asymptotic values, and these will be used to describe the asymptotic geometry of the moduli space in the next section.

For the fermions we have (the ellipsis again indicates terms beyond the large distance expansion)

$$^{(1)}\lambda^\mp_\alpha(x) = -2\,|Q_2|\,g_s^{-3/2}S_{\alpha\beta'}(x, x_0)\,\bar\xi^{\beta'}\left(1 + \dots\right)$$
$$^{(1)}\bar\lambda^\pm_{\beta'}(x) = \pm 2\,|Q_2|\,g_s^{-3/2}\,\xi^\alpha S_{\alpha\beta'}(x, x_0)\left(1 + \dots\right)\,, \tag{9.7}$$

where $S(x, x_0) = -\mathrm{i}\slashed\partial\,G(x, x_0)$ is the $\lambda\bar\lambda$ propagator. The signs in the second equation reflect our choice of conventions for instantons and anti-instantons, see (7.25).

Let us begin with the purely bosonic correlators: With the GCC measure $\mathrm{d}\mu_\xi \equiv \mathrm{d}^2\xi\,\mathrm{d}^2\bar\xi$ $\left(g_s/8|Q_2|\right)^2$ from (8.24) and the Fierz identity $\xi\sigma_\mu\bar\xi\,\xi\sigma_\nu\bar\xi = -\frac12\delta_{\mu\nu}\,\xi\xi\,\bar\xi\bar\xi$, we find in the large distance limit

$$\int\mathrm{d}\mu_\xi\,{}^{(2)}\chi(x)\,{}^{(2)}\chi(y) = \frac{1}{8g_s^2}\,\partial^\mu_0 G(x, x_0)\,\partial^0_\mu G(y, x_0)\,, \tag{9.8}$$

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where $\partial_\mu^0 \equiv \partial/\partial x_0^\mu$ denotes the derivative with respect to the bosonic collective coordinates. Using $^{(2)}\phi = 0$, we then obtain for the leading semiclassical contribution to the correlation function of two scalars in the one-(anti-)instanton background

$$
\begin{aligned}
\langle \phi^A(x)\,\phi^B(y)\rangle_\pm &= g_s^{-2}\,\delta_\chi^A\delta_\chi^B \int \mathrm{d}^4 x_0 \; Y_\pm\,\partial_0^\mu G(x,x_0)\,\partial_\mu^0 G(y,x_0) \\
&= g_s^{-2}\,Y_\pm\,\delta_\chi^A\delta_\chi^B\,G(x,y)\;.
\end{aligned}
\tag{9.9}
$$

Here we denote (remember that the difference between S_{cl} and $S_{\mathrm{inst}}^\pm$ is given by the surface terms (3.11))

$$
Y_\pm \equiv \frac{1}{32\pi^2}\,S_{\mathrm{cl}}^2\,\mathrm{e}^{-S_{\mathrm{inst}}^\pm}K_{\text{1-loop}}^\pm\;,
\tag{9.10}
$$

which is small for small string coupling constant g_s. Since translation invariance implies that neither S_{cl} nor $K_{\text{1-loop}}^\pm$ depend on the collective coordinates x_0, we were allowed to integrate by parts and use $\partial_0^2 G(x,x_0) = -\delta(x-x_0)$. There is no boundary term because the domain of integration covers all of $\mathbb{R}^4$ with no points excised (it is an integral over moduli space, not spacetime), and the integrand vanishes at infinity. The result (9.9) is to be compared with the propagator derived from an effective action with instanton and anti-instanton corrected metric $\mathcal{G}_{AB}^{\mathrm{eff}} = \mathcal{G}_{AB} + \mathcal{G}_{AB}^{\mathrm{inst}}$, with $\mathcal{G}_{AB}$ as in (5.1). Similarly we write for the inverse $\mathcal{G}_{\mathrm{eff}}^{AB} = \mathcal{G}^{AB} + \mathcal{G}_{\mathrm{inst}}^{AB}$, with $\mathcal{G}_{AC}\mathcal{G}^{CB} = \delta_A^B$. At leading order, we find

$$
\mathcal{G}_{\mathrm{inst}}^{AB} = \begin{pmatrix} 0 & 0 \\ 0 & g_s^{-2}(Y_+ + Y_-) \end{pmatrix}\;.
\tag{9.11}
$$

Note that since $Y_- = (Y_+)^*$, instanton and anti-instanton contributions combine into a real correction[10]. This result receives of course corrections from perturbation theory and from terms that become important beyond the large distance approximation in which $\mathrm{e}^{-\phi} \approx g_s^2$. Such terms play a role when inverting the result of (9.11) to obtain the effective metric $\mathcal{G}_{AB}^{\mathrm{eff}}$. They correspond to higher order powers in $Y_\pm$ and interfere with multi-centered (anti-) instanton effects. Dropping all these subleading terms, which is the approximation which we are working in, we find

$$
\mathcal{G}_{AB}^{\mathrm{eff}} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-\phi} - g_s^2(Y_+ + Y_-) \end{pmatrix}\;.
\tag{9.12}
$$

For two RR tensors we find similarly

$$
\int \mathrm{d}\mu_\xi\,{}^{(2)}H_{\mu 1}(x)\,{}^{(2)}H_{\nu 1}(y) = \frac{g_s^2}{8}\,G_{\mu\rho}(x,x_0)\,G_\nu^\rho(y,x_0)\;,
\tag{9.13}
$$

---

[10]We are assuming here that $K^- = (K^+)^*$. Presumably, the one-loop determinants $K^\pm$ only differ by a phase coming from the fermionic determinants. If this phase can be absorbed in the corresponding surface terms (3.11), the instanton and anti-instanton determinants are real and equal.

where $G_{\mu\nu}(x, x_0) = (\partial_\mu \partial_\nu - \delta_{\mu\nu} \partial^2) G(x, x_0)$ is the gauge-invariant propagator of dual tensor field strengths. Using $^{(2)}H_{\mu 2} = 0$ and the convolution property $G_{\mu\rho} * G_\nu^\rho = G_{\mu\nu}$, it follows that

$$\langle H_{\mu I}(x)\, H_{\nu J}(y)\rangle_\pm = g_s^2\, Y_\pm\, \delta_I^1\, \delta_J^1\, G_{\mu\nu}(x, y) \ . \tag{9.14}$$

From the right-hand side we read off the (anti-) instanton correction to the inverse metric M_{IJ}, which multiplies the tensor propagators. We find for the sum

$$M_{IJ}^{\mathrm{inst}} = g_s^2\, (Y_+ + Y_-)\, \delta_I^1\, \delta_J^1\ , \tag{9.15}$$

In the large distance approximation, where also $\chi \approx \chi_\infty$, we then obtain

$$M_{\mathrm{eff}}^{IJ} = M^{IJ} - g_s^{-2}\, (Y_+ + Y_-) \begin{pmatrix} 1 & -\chi_\infty \\ -\chi_\infty & \chi_\infty^2 \end{pmatrix}\ , \tag{9.16}$$

with M^{IJ} as in (5.1). This naively suggests that both RR and NS-NS sectors get corrections in front of the tensor kinetic terms. However, when expressed in terms of $\hat{H}_1 = H_1 - \chi H_2$, the tensor kinetic terms in the effective action simplify to

$$e^{-1}\mathcal{L}_{\mathrm{eff}} = \frac{1}{2}\left(e^\phi - g_s^{-2}(Y_+ + Y_-)\right) \hat{H}_1^\mu \hat{H}_{\mu 1} + \frac{1}{2}\, e^{2\phi} H_2^\mu H_{\mu 2} + \ldots\ , \tag{9.17}$$

In this basis, which is the one to distinguish between fivebrane and membrane instantons (see the discussion in section 2), the NS-NS sector does not receive any instanton corrections.

Last but not least, the mixed bosonic combination

$$\int \mathrm{d}\mu_\xi\, {}^{(2)}H_{\mu 1}(x)\, {}^{(2)}\chi(y) = \mp\frac{1}{8}\, G_{\mu\nu}(x, x_0)\, \partial_0^\nu G(y, x_0) \tag{9.18}$$

obviously vanishes when integrated over x_0 thanks to the Bianchi identity $\partial^\mu G_{\mu\nu} = 0$. We conclude that

$$\langle H_{\mu I}(x)\, \phi^A(y)\rangle = 0\ . \tag{9.19}$$

This was to be expected, since for constant A_A^I the vertex $-\mathrm{i}A_A^I H_I^\mu \partial_\mu \phi^A$ is a total derivative and therefore does not contribute to the propagator. However, in the next section we show that instantons do induce such a vertex with field-dependent coefficients.

3-point functions

While we cannot determine the coefficients A_A^I directly, we can compute the field strength $F_{AB}{}^I = 2\partial_{[A} A_{B]}^I$ from suitable correlation functions and then integrate it. To lowest non-trivial order in an expansion of A_A^I in powers of ϕ^A, the vertex $-\mathrm{i}A_A^I H_I^\mu \partial_\mu \phi^A$ induces a 3-point function

$$\langle \phi^A(x)\, \phi^B(y)\, H_{\mu I}(z)\rangle = \mathrm{i}M_{IJ}^\infty\, \mathcal{G}_\infty^{AC}\, \mathcal{G}_\infty^{BD}\, F_{CD}{}^J \int \mathrm{d}^4 x_0\, G(x, x_0)\, \partial_0^\nu G(y, x_0)\, G_{\mu\nu}(z, x_0)\ . \tag{9.20}$$

The antisymmetric derivative of A_A^I arises here by virtue of the two possible contractions of the scalars and integrating ∂_0^ν by parts. $F_{CD}{}^J$ denotes the constant part of the full field strength. The prefactors M_{IJ}^∞ and $\mathcal{G}_\infty^{AB}$ come from the tensor and scalar propagators, respectively, where the sub- or superscript ∞ indicates replacing the fields by their asymptotic values at infinity. This 3-point function is to be compared with the result of inserting the GCC-dependent fields. Due to the antisymmetry, the two scalars must be different. Since ϕ has no GCC dependence, they have to be contributed by χ and the tensor:

$$\langle \phi(x)\, \chi(y)\, H_{\mu I}(z) \rangle = \delta_I^1 \, \langle \phi_{\rm cl}(x)\, {}^{(2)}\chi(y)\, {}^{(2)}H_{\mu 1}(z) \rangle \ . \tag{9.21}$$

We expand $\phi_{\rm cl}$ as

$$\phi_{\rm cl}(x) = -2\ln(g_s) - \frac{|Q_2|}{g_s^2} G(x, x_0) + \dots \ . \tag{9.22}$$

The leading, constant term reduces the above to the 2-point function (9.18), which vanishes. The next term in the expansion gives, after integrating the GCC,

$$\langle \phi(x)\, \chi(y)\, H_{\mu I}(z) \rangle_\pm = \pm \frac{|Q_2|}{g_s^2} Y_\pm \delta_I^1 \int {\rm d}^4 x_0\, G(x, x_0)\, \partial_0^\nu G(y, x_0)\, G_{\mu\nu}(z, x_0) \ . \tag{9.23}$$

We conclude that

$$\frac{|Q_2|}{g_s^2} (Y_+ - Y_-)\delta_I^1 = {\rm i} M_{IJ}^\infty \, \mathcal{G}_\infty^{\phi A} \, \mathcal{G}_\infty^{\chi B} F_{AB}^{{\rm inst}\,J} \ , \tag{9.24}$$

from which follows

$$F_{\phi\chi}^{{\rm inst}\,1} = -{\rm i}\frac{|Q_2|}{g_s^2}(Y_+ - Y_-) \ , \qquad F_{\phi\chi}^{{\rm inst}\,2} = -\chi_\infty F_{\phi\chi}^{{\rm inst}\,1} \ . \tag{9.25}$$

Next, we compute instanton corrections to $\Gamma^{Ia}{}_b$. It appears in the effective action (4.2) through the relation (C.6), and measures the strength of the coupling between the tensors and the fermions. We therefore compute

$$\langle \lambda_\alpha^a(x)\, \bar\lambda_{\beta'}^{\bar b}(y)\, H_{\mu I}(z) \rangle_\pm = -{\rm i}\frac{|Q_2|}{g_s} Y_\pm \delta_\mp^a \, \delta_\pm^{\bar b}\, \delta_I^1 \int {\rm d}^4 x_0\, \big[S(x, x_0)\bar\sigma^\nu S(y, x_0) \big]_{\alpha\beta'}\, G_{\mu\nu}(z, x_0) \ . \tag{9.26}$$

These two correlators induce an effective vertex $-h_{a\bar a}(\Gamma_{\rm inst}^I)^a{}_b\, \lambda^b \sigma^\mu \bar\lambda^{\bar a} H_{\mu I}$ with coefficients

$$(\Gamma_{\rm inst}^I)^a{}_b = -{\rm i}\frac{|Q_2|}{g_s} M_\infty^{I1}\, (Y_+ \delta_2^a h_{b1} + Y_- \delta_1^a h_{b2}) = -{\rm i}\frac{|Q_2|}{g_s} M_\infty^{I1} \begin{pmatrix} 0 & Y_- \\ Y_+ & 0 \end{pmatrix} \ . \tag{9.27}$$

Here we have used that $h_{a\bar b}$ is not corrected at leading order[11].

---

[11] The 2-point function of two fermion insertions vanishes in the semiclassical limit. Moreover, the result for the 3-point function (9.28) shows that no field dependence of the metric $h_{a\bar b}$ is induced to leading order.

The last two correlators contribute to the connection $\Gamma_A{}^a{}_b$, which appears in the covariant derivative on the fermions (4.3). This connection was zero on tree-level (see (D.5)), but it receives instanton corrections as follows from

$$\langle \lambda_\alpha^a(x)\, \bar{\lambda}_{\beta'}^{\bar{b}}(y)\, \phi^A(z)\rangle_\pm = \mp\mathrm{i}\frac{|Q_2|}{g_s^3}\, Y_\pm \delta_\mp^a\, \delta_\pm^{\bar{b}}\, \delta_\chi^A \int \mathrm{d}^4 x_0 \left[S(x,x_0)\bar{\sigma}^\mu S(y,x_0)\right]_{\alpha\beta'} \partial_\mu\, G(z,x_0)\ . \tag{9.28}$$

This corresponds to an effective vertex $-\mathrm{i}h_{a\bar{a}}(\Gamma_A^{\mathrm{inst}})^a{}_b\, \lambda^b \sigma^\mu \bar{\lambda}^{\bar{a}}\, \partial_\mu\phi^A$ with

$$(\Gamma_A^{\mathrm{inst}})^a{}_b = \frac{|Q_2|}{g_s^3}\, \mathcal{G}_{A\chi}^\infty \begin{pmatrix} 0 & -Y_- \\ Y_+ & 0 \end{pmatrix}\ . \tag{9.29}$$

The above connections induce instanton corrections to the curvature tensors that appear in the four-fermi couplings of the effective action (4.6). Indeed, that these curvatures receive instanton corrections also follows from the computation of 4-point functions of fermionic insertions, and these results should be consistent with computing the curvatures from the connections. For reasons explained in the beginning of this section, checking this consistency may be a complicated task.

Notice, however, that there is another four-fermi term in (4.6) proportional to the product of two antisymmetric tensors $\mathcal{E}_{ab}$. It is easy to see that this tensor cannot receive instanton corrections, since it multiplies only λ^a in the action, not $\bar{\lambda}^{\bar{a}}$. Due to the even distribution of fermionic zero modes among λ^a and $\bar{\lambda}^{\bar{a}}$ there are thus no non-vanishing correlation functions that could induce an effective vertex involving $\mathcal{E}_{ab}$. A similar argument shows that the connections $\Gamma^{Ii}{}_j$ do not get corrected: they occur in the action only in combination with gravitinos (e.g. in the vertex $2\Gamma^{Ii}{}_j H_I^{\mu\nu\rho}\psi_\mu^j\sigma_\nu\bar{\psi}_{\rho i}$, hidden in the square of the supercovariant field strengths of the tensors [29]), which have no zero modes to lowest order in the GCC. Correlation functions of fields corresponding to vertices involving $\Gamma^{Ii}{}_j$ then do not saturate the GCC integrals and vanish. If we were to continue the procedure of sweeping out solutions by applying successive broken supersymmetry transformations to the fields, the gravitinos may obtain a GCC dependence at third order, but then the number of GCCs in the correlators of interest exceeds the number of degrees of freedom and they therefore vanish as well. Note, however, that due to (4.9) the coefficients $\Omega_I{}^i{}_j$ do get corrected:

$$(\Omega_I^{\mathrm{eff}})^i{}_j = M_{IJ}^{\mathrm{eff}}\, \Gamma^{Ji}{}_j = \Omega_I{}^i{}_j + g_s^2(Y_+ + Y_-)\delta_I^1\, \Gamma_{\infty j}^{1i}\ . \tag{9.30}$$

These quantities appear in the supersymmetry transformations of the tensors (4.7).

We can use the results for the various connections in an independent derivation of the field strength components (9.25) by means of the identity (C.5),

$$F_{AB}{}^I = -2\,\mathrm{Tr}\left(W_A^\dagger\, h^t\, \Gamma^I W_B\right) - 2\,\mathrm{Tr}\left(W_A^\dagger\, h^t\, W_B\Gamma^{It}\right)\ , \tag{9.31}$$

which must hold also in the effective theory if supersymmetry is preserved. The first trace on the right contains the connection $\Gamma^{Ia}{}_b$, the second trace the connection $\Gamma^{Ii}{}_j$. As argued above, the latter is not modified by instantons. Using the corrections to the vielbeins derived in appendix F, it is then readily verified that the second trace vanishes identically as a result of $\Gamma^{Ii}{}_j$ being symmetric. On the other hand, the first trace does receive a contribution from the instanton-corrected connection (9.27). A short calculation then yields the same expressions as in (9.25).

Let us now determine the connections A_A^I. Toward this end, we use invariance of the effective action under transformations $A_A^I \to A_A^I + \partial_A \xi^I(\phi)$ to choose a convenient gauge, namely

$$A_\phi^I = 0 \ . \tag{9.32}$$

In this axial gauge we have $F_{\phi\chi}{}^I = \partial_\phi A_\chi^I$. Consider now the fields

$$A_\chi^1 = \mathrm{i}(Y_+ - Y_-) \ , \qquad A_\chi^2 = -\chi_\infty A_\chi^1 \ . \tag{9.33}$$

Using the relation $\partial_{\phi_\infty} = g_s^{-2} \partial/\partial g_s^{-2}$, the derivative with respect to the modulus ϕ_∞ of $Y_\pm$ given in (9.10) yields

$$32\pi^2 \, \partial_{\phi_\infty} \mathrm{i} Y_\pm = -\mathrm{i}\frac{|Q_2|}{g_s^2} \, S_{\mathrm{cl}}^2 \, \mathrm{e}^{-S_{\mathrm{inst}}^\pm} K_{\text{1-loop}}^\pm + 2\mathrm{i}\frac{|Q_2|}{g_s^2} \, S_{\mathrm{cl}} \, \mathrm{e}^{-S_{\mathrm{inst}}^\pm} K_{\text{1-loop}}^\pm$$
$$+ \mathrm{i} S_{\mathrm{cl}}^2 \, \mathrm{e}^{-S_{\mathrm{inst}}^\pm} \, \partial_{\phi_\infty} K_{\text{1-loop}}^\pm \ . \tag{9.34}$$

For small g_s, the second term on the right-hand side is suppressed by a factor g_s^2 as compared to the first term. We can also assume that the derivative of the 1-loop determinant gives only a subleading contribution. In our approximation, the first term is the dominating one and coincides exactly with the field strength (9.25). We conclude that the expressions in (9.32), (9.33) are the sought-after instanton-corrected connection coefficients.

10 The universal hypermultiplet moduli space

In order to determine the instanton corrections to the universal hypermultiplet, we first Wick-rotate back to Lorentz signature and then dualize the tensors H_I into two pseudoscalars $\phi^I = (\varphi, \sigma)$, using the same notation as in the introduction, i.e. φ is a RR field and σ the NS axion. If we combine the latter and $\phi^A = (\phi, \chi)$ into a four-component field $\phi^{\hat{A}} = (\phi^A, \phi^I)$, then in this basis the universal hypermultiplet metric reads [29]

$$G_{\hat{A}\hat{B}} = \begin{pmatrix} \mathcal{G}_{AB} + A_A^I M_{IJ} A_B^J & A_A^K M_{KJ} \\ M_{IK} A_B^K & M_{IJ} \end{pmatrix} \ . \tag{10.1}$$

Using (9.12), (9.15), (9.32) and (9.33), we find for the asymptotic effective Lagrangian

$$
e^{-1}\mathcal{L}_{\text{UH}} = -\frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}\,\mathrm{e}^{-\phi}(1 - g_s^2\mathrm{e}^\phi Y)\left(\partial_\mu\chi\right)^2 - \frac{1}{2}\,\mathrm{e}^{-\phi}(1 + g_s^2\mathrm{e}^\phi Y)\left(\partial_\mu\varphi\right)^2
$$
$$
- \mathrm{e}^{-\phi}\tilde{Y}\,\partial_\mu\chi\,\partial^\mu\varphi - \frac{1}{2}\,\mathrm{e}^{-2\phi}(\partial_\mu\sigma + \chi\partial_\mu\varphi)^2 + \dots \; , \tag{10.2}
$$

where the ellipsis stands for subleading terms. $Y \equiv Y_+ + Y_-$ is the sum of the instanton and anti-instanton contributions, as introduced in (9.10). It can be written as

$$
Y = \frac{1}{32\pi^2}\,S_{\text{cl}}^2\,\mathrm{e}^{-S_{\text{cl}}}\big(\mathrm{e}^{-\mathrm{i}\hat{\sigma}|Q_2|}K_{\text{1-loop}}^+ + \mathrm{e}^{\mathrm{i}\hat{\sigma}|Q_2|}K_{\text{1-loop}}^-\big)
$$
$$
= \frac{1}{16\pi^2}\,S_{\text{cl}}^2\,\mathrm{e}^{-S_{\text{cl}}}K_{\text{1-loop}}\cos(\hat{\sigma}Q_2) \; , \tag{10.3}
$$

where we have introduced $\hat{\sigma} \equiv \sigma + \chi_0\varphi$ such that Y is periodic in $\hat{\sigma}$. The second equality in (10.3) holds only under the assumption made in footnote 10. Similarly, we have

$$
\tilde{Y} \equiv \mathrm{i}(Y_+ - Y_-) = \frac{1}{16\pi^2}\,S_{\text{cl}}^2\,\mathrm{e}^{-S_{\text{cl}}}K_{\text{1-loop}}\sin(\hat{\sigma}|Q_2|) \; . \tag{10.4}
$$

To derive this term[12], we have used that $\chi \approx \chi_\infty$, which holds in the large distance approximation made in this paper. Notice furthermore that only the RR sector receives corrections from the NS5-brane instanton.

The metric and isometries

The next step is to write down the line element, which is given by

$$
\mathrm{d}s_{\text{UH}}^2 = G_{\hat{A}\hat{B}}\,\mathrm{d}\phi^{\hat{A}} \otimes \mathrm{d}\phi^{\hat{B}} \; .
$$

We remind the reader that the classical metric reads

$$
\mathrm{d}s_{\text{UH}}^2 = \mathrm{d}\phi^2 + \mathrm{e}^{-\phi}\mathrm{d}\chi^2 + \mathrm{e}^{-\phi}\mathrm{d}\varphi^2 + \mathrm{e}^{-2\phi}(\mathrm{d}\sigma + \chi\mathrm{d}\varphi)^2 \; , \tag{10.5}
$$

and describes the homogeneous quaternion-Kähler space $\text{SU}(1,2)/\text{U}(2)$ [13]. As mentioned in the introduction, the isometry group $\text{SU}(1,2)$ can be split into three categories. First, there is a Heisenberg subgroup of shift isometries,

$$
\phi \to \phi \; , \qquad \chi \to \chi + \gamma \; , \qquad \varphi \to \varphi + \beta \; , \qquad \sigma \to \sigma - \alpha - \gamma\varphi \; , \tag{10.6}
$$

where α, β, γ are real (finite) parameters. This Heisenberg group is preserved in perturbation theory [16]. We have not discussed these corrections (which only appear at one-loop in the string frame) here; they are discussed in [15, 18] and should be added to our final result for the metric.

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[12]We thank S. Alexandrov for pointing out a mistake in a previous version of this paper.

Second, there is a U(1) symmetry that acts as a rotation on φ and χ, accompanied by a compensating transformation on σ. Its finite transformation can be determined from the results in [12, 28] and reads

$$\varphi \rightarrow \cos\delta \; \varphi + \sin\delta \; \chi \; , \qquad \chi \rightarrow \cos\delta \; \chi - \sin\delta \; \varphi$$
$$\sigma \rightarrow \sigma - \tfrac{1}{4}\sin(2\delta) \; (\chi^2 - \varphi^2) + \sin^2\!\delta \; \chi\varphi \; . \tag{10.7}$$

The remaining four isometries involve non-trivial transformations on the dilaton, and hence will change the string coupling constant. Their infinitesimal form was found in [38, 39], and we will not explore the fate of these isometries nonperturbatively. In fact, at the moment of writing this, it has not been analyzed whether these dilaton-transforming isometries survive the perturbative corrections.

We now present the instanton corrected moduli space metric. As shown above, instanton effects are proportional to Y and $\tilde{Y}$ given by (10.3) and (10.4), and depend on the instanton charge Q_2 and the RR background specified by χ_0. Moreover, also the asymptotic values of the fields, g_s and χ_∞, appear; they are treated as coordinates in the asymptotic regime of the moduli space, i.e., where $\chi = \chi_\infty$ and $\mathrm{e}^{-\phi} = g_s^2$. For fixed values of χ_0 and Q_2, the moduli space metric is given by

$$\mathrm{d}s_{\mathrm{UH}}^2 = \mathrm{d}\phi^2 + \mathrm{e}^{-\phi}(1 - Y)\mathrm{d}\chi^2 + \mathrm{e}^{-\phi}(1 + Y)\mathrm{d}\varphi^2 + 2\mathrm{e}^{-\phi}\tilde{Y}\mathrm{d}\chi\mathrm{d}\varphi + \mathrm{e}^{-2\phi}(\mathrm{d}\sigma + \chi\mathrm{d}\varphi)^2 \; , \tag{10.8}$$

up to subleading terms. This metric therefore satisfies the constraints from quaternionic geometry only up to leading order; to what extent the quaternionic structure can fix these subleading corrections remains to be seen. The result written in (10.8) depends on Q_2 and on the chosen RR background. To obtain the full moduli space metric, one must sum over all instanton numbers Q_2. It would be very interesting to do this sum explicitly, and to see of which function we have the asymptotic limit. Unfortunately, for that we need more knowledge on the one-loop determinants and the subleading corrections.

We can also deduce the leading-order instanton corrections to the vielbeins and other geometric quantities. These can be computed from the vielbeins that determine the double-tensor multiplet geometry, which we give in appendix F.

What happens to the isometries[13] (10.6) and (10.7)? For the Heisenberg group, this amounts to investigating which isometries are broken by the quantities Y and $\tilde{Y}$, as the other terms are invariant. First we focus on the γ-shift in χ. For a given, fixed RR background χ_0, the γ-shift is broken completely. This is because Y is proportional to S_{cl}, which contains $\Delta\chi = \chi_\infty - \chi_0$, see (3.10). However, this symmetry can be restored if we simultaneously change the background by $\chi_0 \rightarrow \chi_0 + \gamma$. Since χ_0 is subject to a quantization

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[13]We assume here that the isometry transformations do not receive any quantum corrections.

condition (see section 3), this induces a quantization condition on the possible values for γ. This means that the γ-shift is broken to a discrete subgroup.

With this in mind, we find that under the action of a generic element in the Heisenberg group the metric is invariant only if the following quantization condition is satisfied:

$$\alpha - (\chi_0 + \gamma)\beta = \frac{2\pi n}{|Q_2|} \ , \tag{10.9}$$

with n an integer. As explained before, the γ-dependence is not relevant here since we could shift the RR background again. For the other two isometries, generated by α and β, only a linear combination is preserved. Stated differently, the β-isometry is preserved as a continuous isometry if we accompany it by a compensating α-shift, where α is determined from (10.9).

If we solely perform an α-transformation, only a discrete $\mathbb{Z}_{|Q_2|}$ subgroup survives as a symmetry. In fact, since the full metric includes a sum over Q_2, only shifts with $\alpha = 2\pi n$ are unbroken. In conclusion, for the Heisenberg group, one isometry remains continuous, and two are broken to discrete subgroups. This is precisely in line with the proposal made in [18].

The remaining isometry we discuss is (10.7). Since the last term in (10.8) is invariant by itself, we should only look at the RR sector. Due to the fact that Y is independent of φ, but depends on χ^2, this continuous rotation symmetry seems to be broken. In fact, the terms proportional to Y and $\tilde{Y}$ break this isometry down to the identity $\delta = 0$ and the discrete transformation with $\delta = \pi$,

$$\chi \to -\chi \ , \qquad \varphi \to -\varphi \ , \qquad \sigma \to \sigma \ . \tag{10.10}$$

This conclusion is different from [12], where also $\delta = \pi/2$ was claimed to survive as an isometry. It is not excluded, however, that in a full treatment the exact answer might restore some of the broken symmetries. Clearly, this is an interesting point that deserves further study.

Notice finally also the existence of another discrete isometry, which changes the sign in χ (or φ) together with a sign flip in σ. This is because the (leading) instanton plus anti-instanton corrections are even in χ and σ. This discrete isometry is however not part of (a discrete subgroup of) SU(1, 2).

11   Conclusions

In this paper we have performed a detailed semiclassical computation of certain correlators in the background of an NS5-brane instanton, in a supergravity theory coupled to the universal hypermultiplet. We have also studied the effects of turning on nontrivial RR

background matter fields. This has resulted in the instanton corrected moduli space for the universal hypermultiplet metric in the asymptotic regime. Our main result can be summarized by (10.8) together with (10.3), and should be combined with the one-loop correction found in [15]. An unexpected result is the fact that the presence of the NS5-brane only affects the Ramond-Ramond sector. Our result for the universal hypermultiplet metric has enabled us to discuss the breaking of isometries, and in particular we have demonstrated the breaking of the Heisenberg group to a discrete subgroup thereof, as explained in the last section, and summarized by (10.9).

Recently, a conjecture for the fivebrane instanton corrections to the universal hypermultiplet moduli space metric was made in [18], using superspace techniques. It would be very interesting to test if this proposal reproduces our results in the semiclassical limit. The structure of the breaking of isometries is already the same in both analyses, but a more detailed comparison is still missing. Vice versa, one should study if the constraints from supersymmetry, namely the quaternionic geometry, are restrictive enough to fix the subleading corrections that we have ignored. Experience from three-dimensional gauge theories on the Coulomb branch suggests that we need some additional information on the regularity and isometry structure of the moduli space.

To make further progress, one clearly needs to embed our calculation into string theory. Perhaps this can shed light on calculating the one-loop determinant in the instanton background, or give more insight into the structure of the subleading terms. Furthermore, from string theory we learn that we should also take into account membrane instanton effects. Just like for fivebranes, membrane instantons have a supergravity description, as was demonstrated in [27, 28]. One can therefore repeat our program for these solutions.

There are several interesting generalizations and applications, of which we mention two. First, one can study the case where more than one hypermultiplet is present. This corresponds to more general Calabi-Yau manifolds with $h_{1,2} \neq 0$. Then, the effective action can also be obtained from a type IIB compactification, or in some cases from the heterotic string on $K3 \times T^2$. In the latter case, one could use the duality with the heterotic string to determine the hypermultiplet moduli space, since on the heterotic side there are no corrections from target space instantons. From the IIB perspective, one could study the consequences of the $SL(2, \mathbb{R})$ symmetry and of nonperturbative mirror symmetry between IIA and IIB. Second, as an application, we would like to compute the instanton corrections to the scalar potentials that are obtained after gauging the unbroken isometries. Perhaps they can lead to new interesting vacua with a nonvanishing cosmological constant. We leave this for further research, and hope to report on this in the future.

Acknowledgments

A Spinor conventions

In this appendix, we elaborate on our conventions and properties of the Euclidean theory. The spinors we work with are the continuation of Lorentzian Weyl spinors λ, $\bar{\lambda}$ which are related by complex conjugation. In Euclidean space, the Lorentz group becomes $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$, and the two-component spinors λ_α and $\bar{\lambda}^{\alpha'}$ belong to inequivalent representations of the two SU(2) factors, not related by complex conjugation. Similarly for the Euclidean supersymmetry transformation parameters, which are labeled by two independent two-component spinors ϵ^i and $\bar{\epsilon}_i$. As a consequence, the Euclidean action is not real, but it is holomorphic in the spinors λ and $\bar{\lambda}$.

The σ^μ and $\bar{\sigma}^\mu$ matrices have lower and upper indices respectively for their matrix entries, and we follow the notation and conventions of Wess and Bagger [40], adapted to Euclidean space,

$$\sigma^\mu = (\vec{\sigma}, -\mathrm{i}) , \qquad \bar{\sigma}^\mu = (-\vec{\sigma}, -\mathrm{i}) , \tag{A.1}$$

consistent with the identification $\sigma^4 = \mathrm{i}\sigma^0$. This implies the properties

$$\sigma^\mu \bar{\sigma}^\nu = -g^{\mu\nu} + 2\sigma^{\mu\nu} , \qquad \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} = \sigma^{\mu\nu} , \tag{A.2}$$

where $\sigma^{\mu\nu} \equiv \frac{1}{2}\sigma^{[\mu}\bar{\sigma}^{\nu]}$, and $\varepsilon^{1234} = 1$ ($= e^{-1}$ in the local case). The second equation is a proper self-duality equation, and differs by a factor of i from the Lorentzian case. We also have

$$\sigma^\mu \bar{\sigma}^\nu \sigma^\rho = g^{\mu\rho}\sigma^\nu - g^{\nu\rho}\sigma^\mu - g^{\mu\nu}\sigma^\rho + \varepsilon^{\mu\nu\rho\sigma}\sigma_\sigma . \tag{A.3}$$

Further properties that are used are

$$(\sigma^\mu)_{\alpha\beta'}\,(\sigma_\mu)_{\gamma\alpha'} = -2\varepsilon_{\alpha\gamma}\,\varepsilon_{\beta'\alpha'} , \qquad (\bar{\sigma}^\mu)^{\beta'\alpha}\,(\sigma_\mu)_{\gamma\alpha'} = -2\delta^\alpha_\gamma\delta^{\beta'}_{\alpha'} , \tag{A.4}$$

from which one can compute

$$(\sigma^{\mu\nu}\varepsilon)_{\alpha\beta}\,(\sigma_\nu)_{\gamma\alpha'} = -\varepsilon_{\gamma(\beta}\,\sigma^\mu_{\alpha)\alpha'} , \tag{A.5}$$

which is the same as in Lorentz signature.

B Dualization and target spaces

As explained in the introduction, the dualization of scalars into tensors changes the geometry of the remaining scalars. For supergravities with maximal supersymmetry, this was demonstrated in [31, 41]. Here, we briefly mention the results for the coset spaces that appear in our model. We distinguish between the Lorentzian and Euclidean theories, since the signs of the pseudoscalars change after Wick rotation. For the universal hypermultiplet, this leads to the following two chains, corresponding to the hypermultiplet, the tensor multiplet, and the double-tensor multiplet respectively. In the Lorentzian theory, the duality chain is

$$\frac{\mathrm{SU}(1,2)}{\mathrm{U}(2)} \longrightarrow \frac{\mathrm{SO}(1,3)}{\mathrm{SO}(3)} \cong \frac{\mathrm{SL}(2,\mathbb{C})}{\mathrm{SU}(2)} \longrightarrow \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{O}(2)} \ . \tag{B.1}$$

Notice that the scalar manifold for the tensor mulitplet is just Euclidean AdS$_3$.

For the Euclidean action, after Wick rotating the scalars, the sigma model metric is no longer positive definite, and we have the duality chain

$$\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(1,1)} \longrightarrow \frac{\mathrm{SO}(2,2)}{\mathrm{SO}(2,1)} \cong \mathrm{SL}(2,\mathbb{R}) \longrightarrow \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{O}(2)} \ . \tag{B.2}$$

The tensor multiplet scalars, the middle step of the chain, now correspond to AdS$_3$.

The geometry of these scalar manifolds must be consistent with the constraints from supersymmetry. These constraints are different in the Lorentzian and Euclidean signatures. For hypermultiplets with Euclidean supersymmetry, the target space is no longer quaternion-Kähler, as already follows from the example given above. The precise constraints on the geometry of the hypermultiplet scalars has, to the best of our knowledge, not been worked out. For the scalars living in $N = 2$ vector multiplets, this was recently done in [42], where it was shown that the usual (special) Kähler geometry is replaced by (special) para-Kähler geometry. We expect that for hypermultiplets the geometry of quaternion-Kähler manifolds will be replaced by the notion of para-quaternionic Kähler geometry. For some mathematics literature on this, see e.g. [43].

C Constraints in the scalar-tensor models

We here present the constraints on and relations between the various quantities appearing in the action (4.2) and supersymmetry transformation rules (4.4) of the scalar-tensor system. These all follow from the requirement of the closure of the supersymmetry algebra and the invariance of the action, and were discussed extensively in [29].

The algebraic relations one finds are

$$\gamma_{ia}^A W_A^{bj} + g_{Iia} f^{Ibj} = \delta_i^j \delta_a^b$$

$$\gamma_{ia}^A \, \bar{W}_{Aj}^{\bar{a}} + g_{Iia} \, \bar{f}^{I\bar{a}}{}_j + (i \leftrightarrow j) = 0 \ , \tag{C.1}$$

for contractions over A and I, and

$$\begin{pmatrix} \gamma_{ia}^A \, W_B^{aj} & \gamma_{ia}^A \, f^{Jaj} \\ g_{Iia} \, W_B^{aj} & g_{Iia} \, f^{Jaj} \end{pmatrix} + \text{c.c.} \, (i \leftrightarrow j) = \delta_i^j \begin{pmatrix} \delta_B^A & 0 \\ 0 & \delta_I^J \end{pmatrix} \ . \tag{C.2}$$

Furthemore, we have

$$\mathcal{G}_{AB} \, \gamma_{ia}^B = h_{a\bar{b}} \, \bar{W}_{Ai}^{\bar{b}} \ , \qquad M^{IJ} g_{Jia} = h_{a\bar{a}} \bar{f}^{I\bar{a}}{}_i \ . \tag{C.3}$$

These relations imply among others that

$$\mathcal{G}_{AB} = h_{a\bar{b}} \, W_A^{ai} \, \bar{W}_{Bi}^{\bar{b}} \ , \qquad M^{IJ} = h_{a\bar{b}} \, f^{Iai} \, \bar{f}^{J\bar{b}}{}_i$$
$$\delta_i^j \, h_{a\bar{b}} = \mathcal{G}_{AB} \, \gamma_{ia}^A \, \bar{\gamma}^{j\,B}_{\bar{b}} + M^{IJ} g_{Iia} \, \bar{g}^j_{J\bar{b}} \ . \tag{C.4}$$

For the complete set of relations, involving the covariant derivatives and curvatures, we refer to [29]. We mention here only some relations useful for later purposes. Of particular interest is the following:

$$F_{AB}{}^I = 2\mathrm{i} M^{IJ} k_{Ja\bar{a}} \bar{W}_{Ai}^{\bar{a}} \, W_B^{ai} - 2 h_{a\bar{a}} \bar{W}_{Ai}^{\bar{a}} \, \Gamma^{Ii}{}_j W_B^{aj} \ , \tag{C.5}$$

where the field strength is defined as $F_{AB}{}^I = 2\partial_{[A} A_{B]}^I$. This field strength also measures the nonvanishing of the covariant derivatives of γ_{ia}^A and W_A^{ai}. The higher order fermion terms in the supersymmetry transformation rules (4.4) contain tensors $\Gamma^{Ia}{}_b$ that satisfy

$$M^{IJ} k_{Ja\bar{a}} = \mathrm{i} h_{b\bar{a}} \Gamma^{Ib}{}_a \ . \tag{C.6}$$

Finally, one can define a covariantly constant tensor

$$\mathcal{E}_{ab} = \tfrac{1}{2} \varepsilon^{ji} \left(\mathcal{G}_{AB} \, \gamma_{ia}^A \, \gamma_{jb}^B + M^{IJ} g_{Iia} \, g_{Jjb} \right) \ , \tag{C.7}$$

which satisfies

$$\mathcal{E}_{ab} \begin{pmatrix} W_A^{bi} \\ f^{Ibi} \end{pmatrix} = \varepsilon^{ij} h_{a\bar{a}} \begin{pmatrix} \bar{W}_{Aj}^{\bar{a}} \\ \bar{f}^{I\bar{a}}{}_j \end{pmatrix} \ . \tag{C.8}$$

This tensor appears explicitly in the four-fermi terms in the supergravity action (4.6).

D   Coefficients for the double-tensor multiplet

The universal double-tensor multiplet provides a solution to the constraints in the previous section. In the following we list the coefficient functions appearing in its classical action and transformation laws which satisfy the relations given above. They receive quantum corrections from instantons, some determined in this paper, and 1-loop effects [15].

For the scalar zweibeins we have

$$\gamma_{ia}^{\phi} = (W_{\phi}^{ai})^{\dagger} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \qquad \gamma_{ia}^{\chi} = \mathrm{e}^{\phi}(W_{\chi}^{ai})^{\dagger} = \frac{1}{\sqrt{2}} \mathrm{e}^{\phi/2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad \text{(D.1)}$$

while the tensor zweibeins are given by, for $I = 1, 2$,

$$g_{1\,ia} = -\frac{\mathrm{i}}{\sqrt{2}} \mathrm{e}^{-\phi} \begin{pmatrix} -\mathrm{e}^{\phi/2} & \chi \\ \chi & \mathrm{e}^{\phi/2} \end{pmatrix} , \qquad g_{2\,ia} = -\frac{\mathrm{i}}{\sqrt{2}} \mathrm{e}^{-\phi} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \text{(D.2)}$$

and

$$f^{1\,ai} = \frac{\mathrm{i}}{\sqrt{2}} \mathrm{e}^{\phi/2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad f^{2\,ai} = \frac{\mathrm{i}}{\sqrt{2}} \mathrm{e}^{\phi/2} \begin{pmatrix} \chi & \mathrm{e}^{\phi/2} \\ \mathrm{e}^{\phi/2} & -\chi \end{pmatrix} . \qquad \text{(D.3)}$$

One may check that these quantities satisfy the relations (C.1)–(C.8), with

$$h_{a\bar{a}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad \mathcal{E}_{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \qquad \text{(D.4)}$$

where we also have taken $\varepsilon^{12} = 1$.

The target space connections for the double-tensor multiplet are particularly simple:

$$\Gamma_A{}^a{}_b = 0 , \qquad \Gamma_\phi{}^i{}_j = 0 , \qquad \Gamma_\chi{}^i{}_j = \frac{1}{2} \mathrm{e}^{-\phi/2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \qquad \text{(D.5)}$$

Since $F_{AB}{}^I = 0$, the scalar zweibeins W_A^{ai}, γ_{ia}^A are covariantly constant with respect to these connections. The tensor $k_{Ia\bar{a}}$ can be determined from (C.6), with

$$\Gamma^{1\,a}{}_b = 0 , \qquad \Gamma^{2\,a}{}_b = -\frac{3\mathrm{i}}{4} \mathrm{e}^{\phi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \qquad \text{(D.6)}$$

Other quantities are the gravitino coefficients in the supersymmetry transformations of the tensors (4.7)

$$\Omega_1{}^i{}_j = \frac{\mathrm{i}}{4} \mathrm{e}^{-\phi} \begin{pmatrix} \chi & 2\,\mathrm{e}^{\phi/2} \\ 2\,\mathrm{e}^{\phi/2} & -\chi \end{pmatrix} , \qquad \Omega_2{}^i{}_j = \frac{\mathrm{i}}{4} \mathrm{e}^{-\phi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad \text{(D.7)}$$

and, by using (4.9), the coefficients in the transformations of the gravitinos

$$\Gamma^{1\,i}{}_j = \frac{\mathrm{i}}{2} \mathrm{e}^{\phi/2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \Gamma^{2\,i}{}_j = -\frac{\mathrm{i}}{4} \mathrm{e}^{\phi/2} \begin{pmatrix} -\mathrm{e}^{\phi/2} & 2\chi \\ 2\chi & \mathrm{e}^{\phi/2} \end{pmatrix} . \qquad \text{(D.8)}$$

Just as in the universal hypermultiplet, the four-λ terms come with field-independent coefficients,

$$\frac{1}{4} V_{ab\,\bar{a}\bar{b}}\, \lambda^a \lambda^b\, \bar{\lambda}^{\bar{a}} \bar{\lambda}^{\bar{b}} = -\frac{3}{8} \left(\lambda^1 \lambda^1\, \bar{\lambda}^1 \bar{\lambda}^1 - 2\, \lambda^1 \lambda^2\, \bar{\lambda}^1 \bar{\lambda}^2 + \lambda^2 \lambda^2\, \bar{\lambda}^2 \bar{\lambda}^2 \right) . \qquad \text{(D.9)}$$

E Zero mode norms

In the computation of zero mode norms we frequently encounter integrals of the form

$$I_p = \int d^4x \, h^{-p} \, (\partial_r h)^2 \, , \qquad h = h_\infty + \frac{Q}{4\pi^2 \, r^2} \, ,$$

for some power $p > 1$. Using $dr \, r^{-3} = -2\pi^2 dh/Q$, this turns into

$$I_p = \left(\frac{Q}{2\pi^2}\right)^2 \text{Vol}(S^3) \int_0^\infty dr \, r^{-3} \, h^{-p} = Q \int_{h_\infty}^\infty dh \, h^{-p} = \frac{Q}{p-1} h_\infty^{1-p} \, . \tag{E.1}$$

The integral diverges for $p \leq 1$.

F Instanton corrections to the vielbeins

Although we are mostly interested in instanton corrections to the scalar and tensor metrics, it is worthwhile also to compute the corrected zweibeins W_A^{ai} etc. Once these are known, one can compute the vierbeins on the quaternionic side by using the results of [29]. Obviously, they are determined only up to $SU(2)$ rotations. We use this fact to choose the components as simple as possible.

We begin with determining W_A^{ai} from the first relation in (C.4), $\mathcal{G}_{AB} = \text{Tr}(W_A^\dagger \, h^t \, W_B)$. Since as noted above both $\mathcal{E}_{ab}$ and $h_{a\bar{a}}$ are given by their classical expressions, (C.8) implies that W_A (and f^I) are of the form

$$W_A^{ai} = \begin{pmatrix} u_A & v_A \\ -\bar{v}_A & \bar{u}_A \end{pmatrix} \, .$$

We solve $\mathcal{G}_{\phi\phi}^{\text{inst}} = \mathcal{G}_{\phi\chi}^{\text{inst}} = 0$ by setting $W_\phi^{\text{inst}} = 0$. Furthermore, we can choose u_χ real and $v_\chi = 0$; then $\mathcal{G}_{\chi\chi}^{\text{inst}} = -g_s^2 Y$ implies

$$W_\chi^{\text{eff} \, ai} = \frac{1}{2\sqrt{2}} \left(2e^{-\phi/2} - g_s Y\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \, . \tag{F.1}$$

The γ^A now follow from (C.3), $\gamma^A = \mathcal{G}^{AB} W_B^\dagger \, h^t$,

$$\gamma_{\text{eff} \, ia}^\phi = \gamma_{ia}^\phi \, , \qquad \gamma_{\text{eff} \, ia}^\chi = \frac{1}{2\sqrt{2}} \left(2e^{\phi/2} + g_s^{-1}Y\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \, . \tag{F.2}$$

The determination of g_I is analogous to the one of W_A; from $M_{IJ} = \text{Tr}(g_I^\dagger \, g_J \, h^{-1t})$ and $M_{12}^{\text{inst}} = M_{22}^{\text{inst}} = 0$ we conclude that $g_2^{\text{inst}} = 0$. For g_1 we take

$$g_{1ia}^{\text{eff}} = g_{1ia} + \frac{i}{2\sqrt{2}} g_s Y \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \, . \tag{F.3}$$

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The coefficients f^I finally are given by the relation $f^I = M^{IJ} g_J^\dagger h^{-1t}$:

$$f_{\text{eff}}^{1ai} = f^{1ai} + \frac{\text{i}}{2\sqrt{2}} g_s^{-1} Y \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \quad f_{\text{eff}}^{2ai} = f^{2ai} - \frac{\text{i}}{2\sqrt{2}} \chi_\infty g_s^{-1} Y \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \quad \text{(F.4)}$$

In section 10 we have observed that the NS sector in the effective action is not affected by instantons. With the above vielbeins we can check whether this might hold for the supersymmetry transformations as well. Indeed, $g_2^{\text{inst}} = 0$ and $\Omega_2^{\text{inst}} = 0$ imply that the transformation (4.7) of the NS 2-form $B_{\mu\nu 2}$, dual to the axion σ, is not corrected, while $\gamma_{\text{inst}}^\phi = 0$ leaves the dilaton transformation unchanged.

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